Hidden Symmetry Unmasked:
Matrix Theory and $E_{11}{=}E_8^{(3)}$

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Abstract

Dimensional reduction of eleven-dimensional supergravity to zero spacetime dimensions is expected to give a theory characterized by the hidden symmetry algebra E_{11}, the end-point of the Cremmer-Julia prediction for the sequence of dimensional reductions of 11d supergravity to spacetime dimensions. In recent work, we have given a prescription for the spacetime reduction of a supergravity-Yang-Mills Lagrangian with large N flavor symmetry such that the local symmetries of the continuum Lagrangian are preserved in the resulting reduced matrix Lagrangian. This new class of reduced matrix models are the basis for a nonperturbative proposal for M theory we have described in hep-th/0408057. The matrix models are also characterized by hidden symmetry algebras in precise analogy with the Cremmer-Julia framework. The rank eleven algebra $E_{11}{=}E_8^{(3)}$ is also known as the very-extension of the finite-dimensional Lie algebra E_8. In an independent stream of work (hep-th/0402140), Peter West has provided evidence which supports the conjecture that M theory has the symmetry algebra E_{11}, showing that it successfully incorporates both the 11d supergravity limit, as well as the 10d type IIA and type IIB supergravities, and inclusive of the full spectrum of Neveu-Schwarz and Dirichlet pbranes. In this topical review, we give a pedagogical account of these recent developments also providing an assessment of the insights that might be gained from linking the algebraic and reduced matrix model perspectives in the search for M theory. Necessary mathematical details are covered starting from the basics in the appendices.

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1 Introduction

In recent works [1, 2], we have presented a nonperturbative proposal for M theory based on a new class of reduced supermatrix models. We give a new prescription for the *spacetime reduction* of a supergravity theory with large N flavor symmetry and in generic curved spacetime background, which preserves the local symmetries of the continuum field theory in the resulting matrix Lagrangian. The models are characterized by extended symmetry algebras reminiscent of the hidden symmetry algebras of dimensionally reduced supergravity theories. We also address in our framework the converse phenomenon, namely, the emergence of a continuum spacetime in the large N limit of the reduced matrix model [2]. In an independent, and very interesting, stream of recent developments [25, 24, 27, 28, 29, 31, 32, 35], West and collaborators have given convincing evidence that the global symmetry algebra of the supergravities with 32 supercharges is E_{11}[2], a result which holds for both 11d supergravity *and* the 10d type IIA and IIB supergravities. E_{11} is further conjectured to be the symmetry algebra of M theory [29, 31]. In this paper, we give a pedagogical review and assessment of some of these developments, stressing the insights into reduced matrix models that might be gained by exploration of the algebraic perspective.

It is well-known that the toroidally-compactified eleven-dimensional supergravity, as well as the ten-dimensional type I-I′, type IIA-IIB, and the heterotic $E_8 \times E_8$ and $Spin(32)/Z_2$, string supergravities, exhibit extended global symmetries as a consequence of the presence of massless scalar fields in the dimensionally-reduced supergravity Lagrangian.[3] In 1978, Cremmer and Julia noticed that the dimensional reduction of a $(D+n)$-dimensional theory containing gravity to D dimensions necessarily results in the appearance of an $SL(n, \mathbf{R})$ global symmetry, as viewed from the perspective of the D-dimensional spacetime [14]. This symmetry is manifest in the form of the dimensionally-reduced Lagrangian. Including an overall scaling of the volume of the compactification manifold, the global symmetry group of the Lagrangian takes the precise form $GL(n, \mathbf{R}) \sim SL(n, \mathbf{R}) \times \mathbf{R}$; the $\mathbf{R}$ factor is, therefore, a *hidden* symmetry of the dimensionally-reduced Lagrangian.[4] Recall that eleven-dimensional supergravity is one of the field theoretic low energy limits of M theory. In [14], Cremmer and Julia conjectured, with partial proof, that the dimensional reduction of 11d supergravity to a Lagrangian in $11-n$ dimensions would result in the appearance of the hidden symmetry group E_n. For $n \geq 3$, this conjecture has since been verified by direct field-theoretic duality transformations on the fields in the dimensionally-reduced classical supergravity Lagrangian [15].

When the $(D+n)$-dimensional gravity theory also contains antisymmetric tensor field strengths,

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[2]Appendix A contains a pedagogical introduction to the infinite-dimensional affine-, over-, and very-extensions of a finite-dimensional Lie algebra. The over and very extensions have a Cartan matrix with Lorentzian signature. Appendix B is a synopsis of the key evidence for E_{11} as the symmetry algebra of theories with 32 supercharges as given in [31].

[3]Dimensional reduction is sometimes distinguished from toroidal compactification by the neglect of the Kaluza-Klein modes. As is conventional in the string literature, we will include all of the massless scalars when identifying the relevant global symmetry group of the theory, irrespective of their origin.

[4]I should perhaps remind the reader at the outset that the n-dimensional Lorentz algebra is contained within $GL(n, \mathbf{R})$, but not within its $SL(n, \mathbf{R})$ subalgebra. However, it is the volume preserving $SL(n, \mathbf{R})$ that is relevant to the spacetime reduction of a field theory to a single spacetime point: it is only in the large N limit of the resulting zero-dimensional matrix model, that both a continuum spacetime, and the full Lorentz algebra, are expected to become manifest. I thank Andrei Mikhailov for requesting this clarification.

dimensional reduction will give rise to axionic scalar fields in D dimensions. A global $\mathbf{R}$ symmetry in a toroidally-compactified supergravity corresponds to a shift symmetry of an axion [14, 15]. As an example, consider the case of eleven-dimensional supergravity with 32 supercharges. Upon dimensional reduction, the metric contributes $(11 - n)$ dilatonic scalars arising from its diagonal component, and $\frac{1}{2}(11-n)(10-n)$ axionic scalars, $\mathcal{A}^i_{[0]j}$. The three-form gauge potential contributes gauge potentials, $(\mathcal{A}_{[3]}, \mathcal{A}_{[2]i}, \mathcal{A}_{[1]ij})$, in D dimensions, in addition to $q=\frac{1}{6}(11-n)(10-n)(9-n)$ axionic scalars, $\mathcal{A}_{[0]ijk}$. Associated with these axions are q shift symmetries, enhancing the global symmetry group to $GL(n, \mathbf{R}) \times \mathbf{R}^q$, where $q = \{0, 0, 0, 1, 4, 10, 20, 35, 56\}$ in $D = \{11, 10, 9, 8, 7, 6, 5, 4, 3\}$. The maximal $\mathbf{R}$ symmetry is realized by all of the new axions in D dimensions that did not exist in $D+1$ dimensions [16]. We emphasize that this conclusion holds *prior* to performing any dualizations: if we were to dualize all axions to antisymmetric tensor gauge potentials, the Lagrangian would only exhibit a $GL(n, \mathbf{R})$ symmetry. In general, there is considerable freedom to alter the precise enlargement of the global symmetry group by invoking appropriate field-dualizations [15]. But we can safely conclude that the volume-preserving factor, $SL(n, \mathbf{R})$, will always be a subgroup of the global symmetry group of the dimensionally-reduced gravity theory. Unlike the case of rigid large N Yang-Mills theories, however, the straightforward dimensional reduction of a locally symmetric theory to $D \leq 2$ dimensions can be fraught with ambiguity in distinguishing scalars and gauge potentials. Thus, a more algebraic perspective on the process of spacetime reduction seems called for [50, 51, 49, 52, 1, 25, 36].

Guided by the observation that there is considerable freedom to alter the hidden symmetry group of a supergravity theory by appropriate field-dualizations [15], it is of interest to ask what precise enhancement of $SL(10, \mathbf{R}) \times G$, where G is the finite-dimensional Yang-Mills group, determines the symmetry group of Matrix Theory? In this paper, we investigate this issue by consolidating insights from the recent works of many authors on the subject of the hidden symmetries of M theory, and of other supergravity theories [15, 26, 17, 24, 31, 34, 35, 36, 55].

If we continue Cremmer and Julia's sequence of dimensional reductions of 11d supergravity to lower dimensions ro its logical endpoint, namely, to *zero* spacetime dimensions, we have the prediction E_{11} for the hidden symmetry group. The notion of spacetime reduction introduced by us in [1] addresses the symmetries of the Lagrangian obtained in this extreme limit: we studied the dimensional reduction of a higher-dimensional supergravity theory to *zero* spacetime dimensions, and also to a single spacetime point. The latter feature was precisely as in Eguchi and Kawai's prescription for planar reduction [3]. The reason our prescription for spacetime reduction recovers a nontrivial large N matrix model, even in the *absence* of Yang-Mills gauge fields in the higher-dimensional continuum field theory, is that our starting point is a supergravity Lagrangian with an additional large N *flavor* group [2].

Planar reduction was first applied to the bosonic rigid large N Yang-Mills theory by Eguchi and Kawai in 1980 [3]. Dimensional reduction of a rigid $U(N)$ Yang-Mills gauge theory to a single spacetime point gives what is known as a reduced unitary matrix model: naively, we set to zero all spacetime derivatives in the Yang-Mills action, retaining the $U(N)$ trace of the square of the commutator of $N \times N$ unitary matrices. The Lagrangian gives a zero-dimensional unitary matrix model with a quartic self-interaction.[5] Reduced matrix models arise, therefore, as the result of

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[5]We use the term *Lagrangian* for a reduced matrix model as follows: the Feynman path integral describing the quantum mechanics of the matrix model is a sum over matrix configurations, weighted by an exponentiated matrix-

a dramatic thinning of the infinite number of degrees of freedom of a quantum field theory upon dimensional reduction of all spacetime fields to a single spacetime point. Remarkably, planar reduced matrix models are found to share many features of exactly solvable unitary matrix models. It should be emphasized that many of the notions familiar from continuum quantum field theory, such as renormalization, universality classes, vacuum structure, and spontaneous symmetry breaking, have their counterpart in the matrix models that follow from spacetime reduction. Likewise, *super*matrix models are obtained when one dimensionally reduces a rigid supersymmetric large N Yang-Mills theory to a spacetime point. Such supermatrix models have been the basis of previous conjectures for nonperturbative string/M theory [5, 6].

With the discovery of Dirichlet-pbranes by Dai, Leigh and Polchinski [9], and with their crucial role as solitonic carriers of dual electric-magnetic charge in the type I and type II string super-gravities clarified by Polchinski in [10], the dimensional reduction of rigid Yang-Mills theories has found an alternative, and rather interesting, new interpretation. Recall that in open and closed string theories, n successive T-duality transformations on n spacetime coordinates parallel to the worldvolume of N coincident D9branes in the type IB string theory carrying 10d nonabelian Yang-Mills gauge fields: $R_n \rightarrow \alpha'/R_n$, where $n \leq 10$, converts n components of the worldvolume gauge bosons to the n components of a scalar field in the n-dimensional spatial bulk orthogonal to the D(9-n)brane [9]. The vacuum expectation values of the n components of the scalar field can be interpreted as the coordinate locations of the D(9-n)brane soliton in an n-dimensional space. In open string theory, this scalar excitation has as vertex operator $(\partial_z X^i_\mu)^2$, where μ=1, $\cdots$, n, and i=1, $\cdots$, N. As first noted by Witten, this implies the tantalizing fact that the "coordinates" of space orthogonal to the N D(9-n)branes arise as the N eigenvalues of n noncommuting, $N \times N$, unitary matrices. For example, with 9 spatial dualizations, we have 9 collective coordinates for the N coincident D0brane solitons: $A^i_\mu(x^0) \leftrightarrow X^i_\mu(x^0)$, $i = 1$, $\cdots$, N, and $\mu = 1$, $\cdots$, 9, where x^0 is time. Here, i is the Chan-Paton index, and the gauge group realized on N coincident D0branes is the nonabelian group $U(N)$, of rank N. In the unoriented type I string theory we obtain, instead, the orthogonal group $SO(2N)$ as worldvolume gauge group [39].

More generally, the X^μ coordinate location of the ith D0brane is the ith eigenvalue of the $U(N)$ matrix X^i_μ, i=1, $\cdots$, N, described above. Restricting to the Yang-Mills field theory on the one-dimensional worldvolume of the D0branes, we have a worldvolume Lagrangian that agrees precisely with the dimensional reduction of the 10d nonabelian Yang-Mills Lagrangian. This gives the familiar quartic interaction for one-dimensional $N \times N$ matrices [9, 11]. Such matrix Hamiltonians describe the quantum mechanics of, time-dependent, large N unitary matrices, as in the Banks-Fischler-Shenker-Susskind proposal for M(atrix) Theory [5]. Planar reduced matrix models, akin to the Ishibashi-Kawai-Kitazawa-Tsuchiya IIB Matrix Model [6], follow as the result of taking this logic one step further: we must T-dualize all *ten* directions of spacetime. The coordinates X^i_μ, with i=1, $\cdots$, N, and $\mu = 0$, $\cdots$, 9, can now be interpreted as the locations of N *Dinstanton events* in a bulk ten-dimensional spacetime. Recall that the tension of a Dinstanton has mass dimension zero. Thus, such a matrix Lagrangian has no dimensionful couplings and is reminiscent of a topological theory.

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valued function. In analogy with field theory, we will refer to the exponentiated function defined on the field of unitary matrices, as the matrix *Lagrangian*. By contrast, in the case of the famous c=1 matrix model, the matrix variables were taken to be functions of an auxiliary parameter later identified with target-space time [4]. Thus, the exponentiated function weighting the Feynman path integral in the case of the c=1 matrix model is the *action*, expressed as a one-dimensional integral over time.

We should emphasize that the spacetime interpretation of reduced unitary matrix model Lagrangians we have just reviewed views the D(9-n)branes as semi-classical solitons in an embedding n-dimensional spacetime. But what dynamical mechanism is responsible for generating the spacetime manifold itself? To address this puzzle, we must delve further into the search for a matrix formulation of M theory. We require a nonperturbative formalism for a fundamental theory of the Universe which addresses the origin of both the long distance interactions *in* an embedding spacetime geometry, as well as the generation of the background geometry itself. Such a theory would capture the full spirit of Einstein gravity: matter and spacetime geometry are set on an equal footing.

It should be noted that both M(atrix) Theory [5], and the IIB Matrix Model [6], are conjectured theories of *induced* gravity: linearized gravity appears as an effective long-distance interaction of the fundamental, pointlike, degrees of freedom, respectively, D0branes or Dinstantons, living in an embedding flat spacetime background. Reconstructing the full nonlinear structure of Einstein gravity from this simplified starting point has proven prohibitively difficult [12], as has the problem of extending the matrix model formalism to curved spacetime geometries [13]. As emphasized by Nicolai [17], there is also no evidence in either matrix model conjecture of the well-established global symmetries of the Einstein supergravities. It was natural to suspect that the dimensional reductions of *locally* supersymmetric Yang Mills theories would be a more relevant direction to explore in the context of conjectures for M theory. Although a concrete suggestion to this effect was made by Nicolai in 1997-98 [17], it appears not to have attracted much attention in the subsequent research literature. Notice, however, that it is not immediately obvious why performing the planar reduction of a locally supersymmetric gauged large N Yang-Mills theory, as in [3], should give a matrix model with nontrivial new large N dynamics. In fact, it will become necessary to modify the Eguchi-Kawai prescription of planar reduction [3] in order to preserve the local symmetries of a gravitational theory with spinors, in generic curved spacetime backgound, in a corresponding reduced matrix model.

In [1, 2], we applied the simple procedure of planar reduction to a supergravity Lagrangian with large N flavor group, and both with, and without, a Yang-Mills gauge sector. We presented our analysis using as prototype the manifestly supersymmetric 10d Lagrangian density obtained in the low energy limit of the heterotic string theory, computed up to quartic order in the α' expansion in [21, 22], and inclusive of gauge-coupling dependent corrections required by closure of the supersymmetry algebra. The resulting planar reduced matrix models with previously studied matrix models, as summarized in Appendix C of this paper. Our discussion includes an especially elegant and simple result for the planar reduction of the 11d supergravity Lagrangian. We explain why simple planar reduction always results in the absence of any remnant of the spectrum of supergravity pform potentials in the corresponding reduced matrix model, despite our introduction of a large N flavor symmetry in order to obtain a nontrivial matrix model.

Appendix C also contains a summary of our modified prescription for *spacetime reduction*, explaining how the local symmetries of the continuum Lagrangian can thereby be preserved in the reduced matrix model. The key insight is to recognize that the Lagrangian density in quantum field theory satisfies locality: thus, the spacetime reduction of all spacetime fields to *linear* forms defined on the infinitesimal patch of local tangent space at a single spacetime point, suffices to preserve all of the local symmetries of the continuum Lagrangian in a corresponding reduced matrix model. We then explain the mechanism for *spacetime emergence* as the eigenvalue coordinates of

the zehn(elf)bein matrix array in the large N limit, demonstrating self-consistency with the basic relations of Riemannian geometry. We exhibit the form of infinitesimal supersymmetry transformations, and of field redefinitions, under spacetime reduction. Notice that the large N flavor symmetry has been chosen to commute with the local symmetries of the Lagrangian, namely, Lorentz, supersymmetry, and gauged Yang-Mills transformations.

In section 3, we review the description of the global symmetry algebra of the nonmaximal 10d supergravity theory with a nontrivial Yang-Mills sector [32], placing it within the larger context of theories with sixteen supercharges [42, 43, 1, 2]. The full details of the precise continuum Lagrangian of interest to us, namely, that of the circle-compactified type I-I′-massive IIA-IIB-heterotic theory, where all six different string theory limits of this theory with sixteen supercharges are obtained by suitable field redefinitions and target-duality transformations alone [37], are as yet unknown. But it is evident that the methodology for the derivation of the relevant reduced supermatrix Lagrangian is clear. The *bosonic* sector of the supergravity Lagrangian of interest to us has been well-studied in the literature, including our previous works [60, 62]. In particular, the full spectrum of supergravity pform potentials has been shown to appear within the Lagrangian framework [37]. Notice that electric, and magnetic dual Dp(6-p)brane pairs, with $-2{\leq}p{\leq}9$, are represented on an equal footing, as in the worldsheet formalism of perturbative string theory [62], and as corroborated by the analysis of the global symmetry algebra of the massive IIA supergravity given by Schnackenburg and West [29], reviewed in section 2 of this paper. Since the self-dual nature of this theory is bound to introduce some subtleties in the form of the full supersymmetric Lagrangian [37, 57], including all of the fermionic terms required by closure of the supersymmetry algebra, as in [21, 22], we do not present a definitive matrix model Lagrangian in this section. Rather, we discuss the important issue of incorporating generic backgrounds, reviewing some established, but less widely-known, facts about the nature of the vacuum landscape of theories with sixteen supercharges. This is the broadbrush picture that has emerged from the detailed study of CHL models [42, 43], and more generic classes of flux compactifications [58, 49, 73] in recent years.

How does our proposal relate to recent studies of the hidden symmetry algebra of M theory? We address this question in Section 4, emphasizing how the algebraic framework can lend significant insight into some key aspects of our proposal. In particular, we present a conjecture for the emergence of theories with 32 supercharges, and no Yang-Mills sector, as a special limit of the theory with sixteen supercharges in the algebraic framework. Concrete evidence for self-duality in the worldsheet formalism of perturbative string theory has been given by us in [64, 60, 62], a work done in partial collaboration with Chen and Novak, building on the earlier results in [63]. We present the worldsheet computation of the tension of a D(-2)braneb coupling to a (-1)form supergravity potential, the magnetic dual of the nine-form potential of massive IIA supergravity. In this paper, we note the corroborating evidence for self-duality presented by Schnakenburg and West in their analysis of the global symmetry algebra of the massive IIA supergravity [28]. We review West's arguments in favor of the very-extended Lorentzian Kac-Moody algebra E_{11} as the hidden symmetry algebra of the ten and eleven dimensional supergravities with 32 supercharges in Appendix B [31]. We conclude with a list of open questions, including those presented in [2], and outline some key directions for future work.

2 Dualizations, Self-duality, and the (-1)form Potential

Let us summarize some of the key insights gained in recent studies of hidden symmetry groups in supergravity. In the original work [14], Cremmer and Julia pointed out that, upon a Weyl rescaling to the Einstein frame metric, the $\mathbf{R}$ subgroup of $GL(n, \mathbf{R})$ becomes a *hidden symmetry*: it is no longer manifest in the Einstein frame Lagrangian. Based on the counting of massless scalar fields in succeeding dimensions, it was conjectured that the hidden symmetry group of the reduced supergravity in $11 - n$ dimensions would take the general form E_{11-n}. The details were worked out for the dimensional reduction to four dimensions, establishing the appearance of the left coset scalar manifold $E_7/SU(8)$. $SU(8)$ is the maximal compact subgroup of E_7 of identical rank. Cremmer and Julia pointed out that the appearance of a coset structure G/H, where G is a noncompact internal symmetry group and H is the compact local invariance group, was a generic consequence of dimensional reduction, implying that the Lagrangian for supergravity scalars always takes the form of a nonlinear realization of a finite semi-simple Lie algebra G [40]. The compact local invariance group H is invariant under the Cartan involution. This expectation has been borne out in subsequent analyses. However, the precise coset form of the hidden symmetry group depends upon performing appropriate dualizations of the fields in the Lagrangian [26]. In dimensions nine and above, there is no enhancement of the $GL(n, \mathbf{R})$ symmetry. In eight dimensions, the $\mathbf{R}_s$ hidden symmetry can be enhanced to an $SL(2, R)$, and the full global symmetry group takes the form $SL(3, \mathbf{R}) \times SL(2, \mathbf{R})$. Likewise, in seven dimensions, $\mathbf{G}_s$ takes the form $SL(5, \mathbf{R})$. Note that, in both cases, the four-form field strength has been dualized. In six dimensions and below, there is a potential clash with the target space duality symmetries of the perturbative string theories [26], so let us turn to that subject.

How does the analysis above relate to the appearance of global symmetry groups in toroidal compactifications of the supersymmetric string theories? The Ramond-Ramond sector's antisymmetric p-form field strengths must now be distinguished from the Neveu-Schwarz sector's symmetric and antisymmetric two-form potentials, g^{ij} and b^{ij}, since the latter can couple directly to the string world-sheet. Thus, if we restrict ourselves to massless scalars arising as perturbative string winding or momentum modes, toroidal compactification of either type II string theory gives the scalar manifold $O(n, n)/[O(n) \times O(n)]$. Likewise, toroidal compactifications of the heterotic string give rise to the scalar manifold $O(n + 16, n)/[O(16 + n) \times O(n)]$. We must also mod out by the T-duality group, respectively, $O(n, n; Z)$, and $O(n + 16, n; Z)$, which corrects for the over-counting of equivalent perturbative string compactifications [39]: there is a stringy $R \to \alpha'/R$ symmetry under the exchange of closed string momentum and winding modes. Finally, recall that the open and closed type I string theory is not T-dual, since open strings can't wind. Thus, in this case, the scalar manifold coincides with what one infers from dimensional reduction. Notice, as has been emphasized by Lu and Pope [26], that no dualizations of NS-NS sector fields are necessary in order to make the full T-duality symmetry manifest in the Lagrangian.[6]

The zero slope limit of the massless IIA string is the same thing as eleven dimensional supergravity compactified on a circle [44, 47, 48]— this is in fact the route by which the IIA Lagrangian was first constructed [45], and so we expect a correspondence in the global symmetry groups in dimen-

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[6]It should be noted that the coset structure of the vacuum manifold can equivalently be inferred from the perspective of the current algebra on the string world-sheet. Toroidal compactifications are isomorphic to (Lorentzian) even self-dual lattices [41, 39]. The CHL moduli spaces are supersymmetry-preserving orbifolds of these [43].

sions nine and below. Including the p additional axions obtained by dualizing all Ramond-Ramond sector field strengths, where $p=(0,0,0,2,4,8,16,32,64)$ in $D=(11,10,9,8,7,6,5,4,3)$ dimensions, the G/H coset takes the form $\mathbf{R}^p \times \{O(n,n;\mathrm{Z})\backslash[O(n,n)/(O(n) \times O(n))]\}$. For $D \geq 7$, the global symmetry group inferred from perturbative type IIA string theory, $\mathbf{R}^p \times O(n,n)$, is a subgroup of the Cremmer-Julia group and there is no clash with T-duality [26]. But it should be noted that the dualization of the R-R four-form field strength was essential in order for this to hold: the T-duality group is not a proper subgroup of the global symmetry group of the fully-undualized dimensionally-reduced Lagrangian in $D \leq 8$ [26]. Moreover, in six dimensions and below, it becomes necessary to also dualize the NS-NS fields in order to enlarge the manifest global symmetries of the Lagrangian beyond the perturbative T-duality symmetry group. $GL(11-n,\mathbf{R})$ and $O(10-n,10-n)$ are both subgroups of E_{11-n}, and their closure indeed generates E_{11-n}. But neither the full $\mathbf{R}^q$ of the fully-undualized global symmetry group, nor the full $\mathbf{R}^p$ following from full R-R dualization, are contained within E_{11-n}!

To settle this ambiguity, it is incumbent upon us to understand the significance of alternative dualizations from a more physical standpoint. Consider the table below summarizing the results of various possible dualizations which we reproduce from the paper of Lu and Pope [26]. We will also note their observation that the global symmetry group of the eleven-dimensional supermembrane, $GL(11,\mathbf{R}) \times R^p$, contains the perturbative T-duality group of the massless IIA string, $O(10-n,10-n)$, as a proper subgroup, while $GL(10,\mathbf{R})$ does not. Finally, in two dimensions and below, the situation gets even murkier. It is tempting to continue the Cremmer-Julia conjecture, arguing for the appearance of the affine extension of the finite-dimensional Lie algebra E_8, namely, E_9, in $D=2$, the first of the hyperbolic Kac-Moody algebras, E_{10}, in $D=1$, and the non-hyperbolic, Lorentzian Kac-Moody algebra, E_{11}, in $D=0$ [24]. Julia had already conjectured the appearance of E_9 and E_{10} back in 1981 [50]. But the entire framework of dimensional reduction and of duality transformations breaks down in this regime, essentially because a scalar field can no longer be sensibly distinguished from a gauge potential in two dimensions.

	Global Symmetry Groups			T-duality
D	No dualisation	R-R dualisation	Full dualisation	
9	$GL(2,R)$	$GL(2,R)$	$GL(2,R)$	–
8	$R * GL(3,R)$	$SL(3,R) \times SL(2,R)$	$SL(3,R) \times SL(2,R)$	$O(2,2)$
7	$R^4 * GL(4,R)$	SL(5,R)	$SL(5,R)$	$O(3,3)$
6	$R^{10} * GL(5,R)$	$R^8 * O(4,4)$	$SO(5,5)$	$O(4,4)$
5	$R^{20} * GL(6,R)$	$R^{16} * O(5,5)$	$E_{6(6)}$	$O(5,5)$
4	$R^{35} * GL(7,R)$	$R^{32} * O(6,6)$	$E_{7(7)}$	$O(6,6)$
3	$R^{56} * GL(8,R)$	$R^{64} * O(7,7)$	$E_{8(8)}$	$O(7,7)$

Table 1: Global Symmetry Groups for Supergravities with 32 Supercharges in D$\geq$3

It should be noted that there is no difficulty in correctly identifying the perturbative T-duality group of compactified string theories in dimensions less than 3. The reason is that the world-sheet current algebra and the equivalent characterization by Lorentzian self-dual lattices, or orbifolds thereof, continue to be perfectly good tools for identifying the global symmetry group even when

$D \leq 2$. In fact, both toroidal, and supersymmetry-preserving orbifold, compactifications of the heterotic and type II string theories to two dimensions were widely explored by Chaudhuri and Lowe in [49]. The basic message is that it is helpful to shift focus to purely algebraic techniques in lower dimensions. Indeed, the recent elucidation of an E_9 affine Lie algebra in two dimensions in [51, 52] was based on Nicolai's 1987 reformulation of eleven-dimensional supergravity, replacing the Lorentz $SO(1, 10)$ group with $SO(1, 2) \times SO(16)$ [51]. Note that only an $SO(1, 2) \times SO(8)$ subgroup of the eleven-dimensional Lorentz group was preserved in this reformulation: the Cremmer-Julia symmetry group does *not* follow from straightforward dimensional reduction in dimensions $D \leq 2$.

Let us now return to the ambiguity in the hidden symmetry group as a consequence of alternative dualizations in $D \leq 6$. A hint in the right direction is provided by Roman's ten-dimensional type IIA cosmological constant [38], later identified by Polchinski as the D8brane charge of the type IIA string theory [10]. We will find that the vexing problem of accommodating a generator corresponding to a nine-form gauge potential in the hidden symmetry algebra of M theory leads to the remarkable conclusion that the Cremmer-Julia E_{11-n} symmetries are not simply one of many options. Rather, they are *required* by the necessity of incorporating both the D8brane and its magnetic dual. The evidence pointing to this conclusion comes largely from the interesting recent works of Peter West [25, 24, 28, 29, 31].

We begin with a brief review of work on the M theory origin of the D8brane, which has been a long-standing puzzle. Roman's original construction introduced the mass parameter as a deformation of the field equations of the ten-dimensional IIA supergravity. Subsequently, Bergshoeff, Green, Hull, Papadopoulos, and Townsend [37] showed that, with suitable field redefinitions, there exists a form of the covariant Lagrangian with the mass parameter, M, appearing explicitly as an auxiliary field. Taking the limit $M \to 0$ smoothly recovers the massless IIA supergravity Lagrangian. The field equation for the auxiliary M simply sets the ten-form field strength equal to a constant $\times$ the epsilon symbol. The explicit appearance of a nine-form gauge potential in the Lagrangian clarifies how it couples to an D8brane, but raises the question of its magnetic dual. Formally, this is a (-1)-form gauge potential with scalar field strength which should couple to a purported D(-2)brane. It follows that the D8brane is potentially a problem for any formalism based on the notion of self-duality that makes explicit use of gauge potentials [56, 57, 59].

For example, a new formalism for supergravity which extends the coset space description of the scalars to the p-form gauge fields was proposed in [57] based on doubled fields. This has the suggestive consequence that the equations of motion are elegantly formulated as a self-duality condition on the total field strength, which is a Lie superalgebra-valued object [50, 57]. However, the nine-form potential has been left out of this discussion. When suitably incorporated, as was shown by [59], self-duality forces one to include consideration of a (-1)-form gauge potential. The tension of the associated D(-2)brane fits neatly into the tower of *jade relations* derived in [59]: equalities relating the tensions of the various branes in the duality spectrum. It is natural to ask how one might sensibly accommodate the notion of a (-1)-form potential and its associated D(-2)brane.

A partial answer to this question is provided by the worldsheet formalism of type I′ string theory. Using the covariant string theory path integral [60, 62], we have shown that the tension of the magnetic dual of the D8brane can be calculated from first principles. Recall that Polchinski's Dpbrane tension calculation covered the range $-1 \leq p \leq 9$ [10]. The brane-tension was extracted from the factorization limit of the one-loop open string amplitude with boundaries on parallel and

static Dpbranes: the end points of the open string lie in a $(p+1)$-dimensional worldvolume. It turns out that the one-loop amplitude calculated in [10] permits precisely one possible generalization from the perspective of two-dimensional Riemannian geometry. This is the one-loop amplitude with *fixed* boundaries, the formalism for which was developed in the earlier works [63, 64]. The factorization limit of the amplitude with fixed boundaries yields the tension of an additional Dbrane. Remarkably, the result obtained for the tension matches perfectly with Polchinski's generic result if we set $p = -2$. The world-sheet formalism of string theory has no difficulty accommodating *both* a D8brane *and* its magnetic dual, the so-called D(-2)brane, in the duality spectrum.

An elegant and simple explanation of the so-called (-1)-form potential that lends credence to our result appears in a work by Schnakenburg and West [28]. The formalism of doubled fields [57] did not clarify how the coset unification of scalars and gauge fields might be extended to also incorporate the metric and fermionic fields of supergravity theories. An alternative approach was subsequently proposed by West [25], in which the entire bosonic sector of both eleven-dimensional supergravity, and of the massless type IIA and type IIB supergravities [27], were formulated as coset non-linear realizations of an appropriate Kac-Moody superalgebra. A key observation made by Cremmer, Julia, Lu, and Pope [15] highlighted in [25, 24], was to note the one-to-one correspondence between the massless fields of the bosonic sector of a supergravity theory, and the nodes in the Dynkin diagram of an E_{11-n} algebra. This leads irrefutably to the conclusion that the dimensional reduction of eleven-dimensional supergravity to zero dimensions in the zero volume limit *must* result in a rank 11 Kac-Moody algebra. The question that remains is which specific algebra.

A hint pointing towards E_{11} is the successful identification of generators corresponding to the nine-form gauge potential, *and even its magnetic dual*[28]. The key observation is that the momentum generator, P_a, already plays the role of the generator corresponding to a putative "(-1)-form" gauge potential in the hidden symmetry algebra of the massive type II supergravity: there is no need to invoke an additional spacetime field representing the "(-1)-form" generator in the non-linear realization. Thus, the complete bosonic field content of the massive IIA theory is simply:

$$h_a^b, A, A_c, A_{c_1 c_2}, A_{c_1 c_2 c_3}, A_{c_1 \cdots c_5}, A_{c_1 \cdots c_6}, A_{c_1 \cdots c_7}, A_{c_1 \cdots c_8}, A_{c_1 \cdots c_9} \quad . \tag{1}$$

A is the dilaton, and $A_{c_1 c_2}$ is the Neveu-Schwarz twoform potential, coupling to perturbative type IIA closed strings. Their Hodge duals are the eight-form, and six-form, gauge potentials, respectively. The 1-form and 3-form gauge potentials, and their Hodge dual 5-form and 7-form potentials, are from the Ramond-Ramond sector. No dual fields have been introduced for the nine-form potential, nor for the metric. We now introduce generators, $R^{c_1 \cdots c_p}$, corresponding to each p-form gauge field listed in Eq. (1). Let us denote the generators of $GL(10, \mathbf{R})$ as K_a^b, then $GL(10, \mathbf{R})$ invariance manifests itself in the commutation relations:

$$[K_b^a, K_d^c] = \delta_b^c K_d^a - \delta_d^a K_b^c, \qquad [K_b^a, P_c] = \delta_c^a P_b, \qquad [K_b^a, R^{c_1 \cdots c_p}] = \delta_b^{c_1} R^{ac_2 \cdots c_p} + \cdots \quad . \tag{2}$$

The difference of the K_b^a are the generators, J_b^a, of $SO(9, 1)$ Lorentz transformations. Including the generators of translations, P_a, we have the additional commutation relations:

$$[R, P_a] = mb_0 P_a, \qquad [P_a, R^{c_1 \cdots c_q}] = -mb_q(\delta_a^{c_1} R^{c_2 \cdots c_q} + \cdots) \quad . \tag{3}$$

If we now check the commutation relations among the $R^{c_1 \cdots c_p}$ by themselves, we find that Eq. (3) is simply a special case of this algebra with P_a playing the role of a putative (-1)-form potential [28]:

$$[R, R^{c_1 \cdots c_p}] = c_p R^{c_1 \cdots c_p}, \qquad [R^{c_1 \cdots c_p}, R^{c_1 \cdots c_q}] = c_{p,q} R^{c_1 \cdots c_{p+q}} \quad . \tag{4}$$

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We can set $c_{-1} = mb_0$. As pointed out in [28], the limit $m \to 0$ smoothly recovers the hidden symmetry algebra of the massless type IIA supergravity, denoted by $\mathcal{G}_{\text{IIA}}$ in [27]. Notice the satisfying agreement with the physical interpretation of the (-1)-form potential in the corresponding worldsheet amplitude with fixed boundaries [60, 64]: the momentum generator acts as a derivative operator, *removing* a worldvolume of codimension one.

The structure constants in Eq. (4) can be determined by verifying consistency with the Jacobi identities [28]:

$$c_2 = -c_6 = \frac{1}{2}, \quad c_3 = c_5 = -\frac{1}{4}, \quad c_1 = -c_{-7} = -\frac{3}{4}, \quad c_{-1} = -c_9 = -\frac{5}{4}$$
$$c_{1,2} = -c_{2,3} = -c_{3,3} = c_{1,5} = c_{2,5} = 2, \quad c_{3,5} = 1, \quad c_{2,6} = 2, \quad c_{1,7} = 3$$
$$c_{2,7} = -4, \quad b_2 = -\frac{1}{2}, \quad b_7 = -\frac{1}{2}, \quad b_0 = \frac{5}{8} \quad . \tag{5}$$

Consistency with the equations of motion of massive type IIA supergravity, and validity of the proposed non-linear realization of the hidden symmetry algebra, have been assumed in deriving Eq. (5).[7] As was shown by Schnakenburg and West, the analysis above can be adapted to the type IIB theory [27], with algebra denoted as $\mathcal{G}_{\text{IIB}}$. The IIB theory has doublets of zero-form and two form fields, namely, the dilaton and axion, and the NS and RR sector two-form potentials with, respectively, eight-form, and six-form, Hodge dual doublets. No Hodge dual is introduced for the four-form RR potential, as in [57, 25], since its field strength satisfies a self-duality condition: the introduction of a "double" for a self-dual field would simply create a mismatch in the number of physical degrees of freedom. This mismatch is reflected in the well-known impossibility of writing down a covariant Lagrangian for the IIB supergravity theory [65].[8] The generators of the hidden symmetry algebra are [27]:

$$K_b^a, P^a, R_s, R_s^{c_1 c_2}, R^{c_1 c_2 c_3 c_4}, R_s^{c_1 \cdots c_6}, R_s^{c_1 \cdots c_8}, \qquad s = 1, \ 2 \quad , \tag{6}$$

where $s = 1$, 2 distinguish potentials in the NS-NS, R-R, sectors, respectively. The generators satisfy the commutation relations given in Eq. (2), as well as the new relations:

$$[R_1, R_s^{c_1 \cdots c_p}] = d_p^s R_s^{c_1 \cdots c_p}, \quad [R_2, R_{s_1}^{c_1 \cdots c_p}] = \tilde{d}_p^{s_2} R_{\epsilon^{s_1 s_2 s_2}}^{c_1 \cdots c_p}, \quad [R_{s_1}^{c_1 \cdots c_p}, R_{s_2}^{c_1 \cdots c_q}] = c_{p,q}^{s_1, s_2} R_{\epsilon_{s_1 s_2 s_2}}^{c_1 \cdots c_{p+q}} \quad . \tag{7}$$

Notice that the dilaton and axion differ in their action on the remaining p-forms, respectively, acting so as to preserve, or switch, the generators within a doublet. Recall that, unlike the IIA algebra, the spinors of the IIB supergravity have identical chirality, precluding the possibility of a mass parameter in the supersymmetry algebra [38]. Thus, unlike the previous case, there is no non-trivial extension of the global algebra of p-form generators by the momentum generator [27]. The structure constants of this algebra are determined by requiring consistency with the Jacobi identities [27]:

$$d_{p+1}^1 = -\tfrac{1}{4}(p-3), \quad d_{p+1}^2 = \tfrac{1}{4}(p-3), \quad \tilde{d}_2^1 = -\tilde{d}_6^2 = -\tilde{d}_8^2 = 1, \quad \tilde{d}_2^2 = \tilde{d}_6^1 = \tilde{d}_8^1 = 0$$
$$c_{2,2}^{1,2} = -c_{2,2}^{2,1} = -1, \quad c_{2,4}^{2,2} = -c_{2,4}^{2,1} = 4, \quad c_{2,6}^{1,2} = 1, \quad c_{2,6}^{1,1} = -c_{2,6}^{2,2} = \tfrac{1}{2} \quad . \tag{8}$$

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[7]I thank P. West for email clarification of this point.

[8]A clear explanation of some subtleties in quantizing a self-dual field strength appears in [56].

How are the global symmetry algebras of the type II supergravities related to $E_{11} = E_8^{(3)}$? This question has been addressed in detail in the very recent paper by West [31], clarifying the precise relationship of $\mathcal{G}_{\mathrm{IIA}}$, $\mathcal{G}_{\mathrm{mIIA}}$, and $\mathcal{G}_{\mathrm{IIB}}$, to $E_8^{(3)}$. It is remarkable that the global symmetries of each of the IIA, IIB, and massive IIA supergravities, as well as those of a broad spectrum of well-known solutions of the classical supergravities inclusive of the full spectrum of Dbranes and Mbranes can be elegantly unified within $E_8^{(3)}$ [31]. In Appendix B, we review West's arguments in more detail, invoking the framework of nonlinear realizations [14, 15, 24, 31].

3   The Theory with Sixteen Supercharges

In Appendix C, we have reviewed the detailed prescription for the spacetime reduction of a locally supersymmetric theory with large N flavor group to a single point in spacetime given by us in [2], such that the resulting zero-dimensional large N matrix model Lagrangian manifests all of the local symmetries of the original continuum field theory. Our prescription is a modification of Eguchi and Kawai's well-known planar reduction procedure, which takes into account the necessity for an auxilliary local tangent space in a covariant Lagrangian formulation of a gravitational theory describing spinors in a generic curved spacetime background. As our prototype example, we have analyzed the case of the heterotic 10d N=1 supergravity-Yang-Mills Lagrangian with an anomaly-free Yang-Mills gauge group, $E_8 \times E_8$ or $SO(32)$, of rank 16. In part, the reason for this is that a detailed analysis of the low energy spacetime effective action, up to quartic order in the inverse string tension, α', and in the inverse Yang-Mills coupling as required by closure under supersymmetry, exists for the heterotic string supergravity [22]. This comprehensive analysis is due to Bergshoeff and Roo [21, 22], building on the earlier works of [19, 20]. The resulting Lagrangian has been presented in manifestly supersymmetric form, and in terms of component fields. The equivalence of the dual two-form and six-form formulations, at least up to quartic order in α', has also been established by these authors. Partial comparisons have been made, and are in agreement with, terms in the effective action inferred from direct string amplitude calculations up to one-loop order [23].

Following the c.1995 developments in string duality, we have an enhanced appreciation of the rich structure of the vacuum landscape of theories with sixteen supercharges. Toroidal compactification of the 10d heterotic string preserves all of its supersymmetries, yielding a rich class of theories with sixteen supercharges, and anomaly-free rank $16+n$ Yang-Mills gauge group, in $10-n$ spacetime dimensions [41]. The discovery of the CHL moduli spaces [42] clarified that the vacuum landscape is *not* simply-connected: these models are supersymmetry preserving orbifolds of the standard toroidal compactifications of the heterotic string [43]. Thus, for example, in nine spacetime dimensions the vacuum structure of the theory with sixteen supercharges is already multiply connected: in addition to the connected vacuum landscape with 17 abelian one-forms at generic points in the moduli space, we have an isolated island universe with 17-8=9 abelian one-forms at generic points.[9] This

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[9]Since the orbifold twist becomes trivial in the noncompact decompactification limit where the requisite massless gauge bosons are simultaneously recovered, the CHL orbifold is not, strictly speaking, a disconnected component of the theory with 16 supercharges [43]. But we should emphasize that, at weak coupling, and in the moduli space approximation, each moduli space describes low-energy physics in a different island universe. While nonperturbative dynamics can often be invoked to infer the possibility of tunneling to a different moduli space with *fewer* supersym-

theory was first identified as an asymmetric orbifold of the circle compactification of the $E_8 \times E_8$ heterotic string theory by Chaudhuri and Polchinski [43]: the Z_2 orbifold action is a supersymmetry-preserving shift in the one-dimensional momentum lattice, accompanied by the outer automorphism exchanging the two E_8 lattices. The gauge symmetry at generic points in the moduli space is rank 9.

We emphasize that there is no known spacetime dynamics, field-theoretic or string-theoretic, that can repair the disconnectedness of the moduli space with sixteen supercharges. Recall that there is no Higgs mechanism in theories with sixteen supercharges. Thus, while the precise enhanced gauge group can vary from point to point, the rank of the abelian subgroup is *fixed* for all points in a connected component of the moduli space [42]. More precisely, as is clarified by the orbifold construction [42, 43, 49], each isolated component of the moduli space is characterized by a distinct target-space duality group entering into specification of the global symmetry algebra of that island universe. An alternative viewpoint is to realize that each island universe is an example of a flux compactification [58]: one, or more, of the supergravity pform fluxes is nontrivial, an invariant on a connected component of the moduli space. The type IIA string duals of the heterotic CHL models with nontrivial Ramond-Ramond one-form flux constructed by Chaudhuri and Lowe [49] were the earliest known examples of flux compactifications of the type II string theory. While the notion of isolated universes can be disconcerting, raising the spectre of the anthropic principle, and banishing hopes of a unique vacuum state for String/M theory picked by dynamics alone, we have argued elsewhere that the problem could be one of misinterpretation [62].

Let us move on to a different aspect of the vacuum landscape of theories with sixteen super-charges, namely, the fact that the six different string theories: type I, type I$'$, type IIA, type IIB, heterotic $E_8 \times E_8$, and heterotic $SO(32)$, each describe a different weakly-coupled limit of the *same* moduli space. Consider the circle compactifications of all six string supergravities and, for convenience, let us restrict ourselves to discussion of the standard component of the moduli space characterized by a rank 16 anomaly-free Yang-Mills gauge group. As is well-known, the Lagrangian we have described above can be mapped by a strong-weak coupling duality transformation, and suitable field identifications, into that of the type IB string theory [70]. Thus, the $SO(32)$ type I string theory is the strong-coupling dual of the heterotic string theory with identical gauge group.

In nine dimensions, and below, the $SO(32)$ and $E_8 \times E_8$ heterotic string theories are related by a target space duality: $R_9 \leftrightarrow \alpha'/R_9$. What is the type I strong-coupling dual of heterotic vacua with states in the spinor representations of the orthogonal groups, as required by the appearance of exceptional Lie algebras? Fortunately, upon compactification to nine dimensions, the type I theory can acquire nonabelian gauge symmetries of nonperturbative origin. Under the T_9 duality, the type I string with its 32 D9branes is mapped to a type I$'$ vacuum with 32 D8branes: additional massless gauge bosons can arise from the zero length limit of D0-D8brane strings. Such D0-D8brane backgrounds preserve all sixteen supersymmetries. The incorporation of D0-D8backgrounds, in addition to those with only 32 D8branes, permits identification of type I-I$'$ strong-weak coupling duals for *all* of the nine-dimensional ground states of the heterotic string theories. In particular, this includes compactifications on a circle of both the $Spin(32)/Z_2$ and the $E_8 \times E_8$ heterotic string

metries [73], no such examples are known in theories with 32 or 16 supercharges. Note that the mechanism proposed for a partial breaking of supersymmetry described in [77], giving a theory with 12 supercharges, requires assumptions about the nature of the theory in the decompactification limit.

theories [70, 72, 60]. the inclusion of the D0-D8brane backgrounds also enables the identification of the type I-I$'$ strong coupling duals for all of the heterotic CHL moduli spaces with sixteen supersymmetries [42, 43, 49]. Furthermore, since type I$'$ theory compactified on S^1 is the same thing as M theory compactified on $S^1 \times S^1/Z_2$, these observations are consistent with the identification of M theory on S^1/Z_2 as the strong coupling limit of the $E_8 \times E_8$ heterotic string theory [47, 71].

Most importantly, the successes in unifying the circle-compactified heterotic-typeI-I$'$ theories with sixteen supercharges can be extended to incorporate the type IIA and type IIB string theories. There exists a nine-dimensional Lagrangian formulation of the massive type IIA-IIB string theories due to Bergshoeff, de Roo, Green, Papadopoulos, and Townsend [37] which incorporates the full spectrum of Dbrane p-form potentials [10], including Roman's IIA cosmological constant [38]. By combining field-dualizations, as well as S and T-duality transformations on the couplings, this Lagrangian can be mapped to *any* of six supergravity theories: the circle-compactified type I, type IIA, or heterotic string supergravities, the Scherck-Schwarz reduction of the type IIB string supergravity, or the $S^1 \times S^1/Z_2$ compactification of eleven-dimensional supergravity. This covers all six vertices of a modified star diagram linking theories with sixteen supercharges [47, 39, 1].

With our new prescription for spacetime reduction, we have shown that the local symmetries of the supergravity Lagrangian can be preserved in the reduced matrix model. Thus, there is a precise analog for each field redefinition, or dualization, of the continuum Lagrangian in the matrix model: the matrix Lagrangian is only unique upto appropriate dualizations defined on the matrix variables [1]. On the one hand, this is a beautiful illustration of the unity of the different string theories with eleven-dimensional supergravity. But it points to the importance of understanding the global symmetry algebra: the identification of a specific, hidden symmetry algebra is what gives precise meaning to one, or other, class of supergravity/M theory toroidal compactifications. This observation has been reiterated recently by West [29, 28, 31, 32]. But it is not new to string theory, nor to supergravity: target-space duality groups, and their conjectured extension to U-dualities [47], have been the bulwark of our understanding of string and M theory compactifications. In section 5 of this paper, we will argue that the notion of the global symmetry algebra also provides a precise generalization incorporating *all* of the ground states of M theory, beyond toroidal compactifications. Remarkably, this will include the isolated island universes discovered in [42].

We began this section by pointing out that, at the current time, we do not have a comprehensive analysis of the full covariant Lagrangian— including all of the fermionic contributions necessitated by supersymmetry, for any of the low energy effective Lagrangians other than that of the heterotic string theory [21, 22]. We save this for future work. The important puzzle of how one might incorporate theories with 32 supercharges, and no Yang-Mills fields, as a special limit of the vacuum landscape of theories with sixteen supercharges will be addressed in the next section.

4   M Theory and its Hidden Symmetry Algebra

We have alluded earlier to the existence of a hidden symmetry algebra in the matrix Lagrangian that is larger than the obvious $U(N) \times G$. In part, there is an $SL(10, \mathbf{R})$ symmetry, which is the manifest remnant under spacetime reduction to a single point in spacetime of the Lorentz symmetry group of the 10d continuum field theory Lagrangian. In the Introduction, we have already explained

the simple rationale for expecting the symmetry algebra of eleven-dimensional supergravity, and consequently, of M theory, to be $E_{11} = E_8^{(3)}$, the rank eleven algebra known as the very-extension of the finite dimensional Lie algebra E_8 [31]. How does the symmetry algebra of the theory of sixteen supercharges relate to that of M theory? We address that question in this section, following the discussion given in [2].

The nonmaximal 10d supergravity is a theory with sixteen supercharges. In the notation of [31], we have the following symmetry generators:

$$K_b^a, R, R^{c_1 c_2}, R^{c_1 \cdots c_6}, R^{c_1 \cdots c_8} \quad . \tag{9}$$

The heterotic supergravity theory has zero-form dilaton and NS two-form potentials, plus their ten-dimensional Hodge duals, respectively, six-form and eight-form supergravity potentials. The K_b^a are the generators of $GL(10, \mathbf{R})$ in the notation of [31]. The commutator algebra of these generators was given in [32]. Not surprisingly, we will find that it agrees precisely with the algebra that can be inferred from the appropriate projection on the global symmetry algebra of the type IIB supergravity. This reflects the well-known connection between these two string theories under the orientation projection identifying left-moving and right-moving modes on the worldsheet [39].

The global symmetry algebra of the type I supergravity can be identified by taking an appropriate projection of the global symmetry algebra, $\mathcal{G}_{IIB}$, of the 10d type IIB supergravity, which has been obtained in a recent work of Schnakenburg and West [29]. By setting the extra forms to zero in Eqs. (1.1-.3) of [29], we find the usual $GL(10, \mathbf{R})$ algebra, extended by translations:

$$[K_b^a, K_d^c] = \delta_b^c K_d^a - \delta_d^a K_b^c, \qquad [K_b^a, P_c] = \delta_c^a P_b, \qquad [K_b^a, R^{c_1 \cdots c_p}] = \delta_b^{c_1} R^{a c_2 \cdots c_p} + \cdots \quad , \tag{10}$$

plus the simplified algebra of 0, 2, 6, and 8-form generators:

$$[R, R^{c_1 \cdots c_p}] = d_p R^{c_1 \cdots c_p}, \quad [R^{c_1 \cdots c_p}, R^{c_1 \cdots c_q}] = c_{p,q} R^{c_1 \cdots c_{p+q}} \quad . \tag{11}$$

Comparing with the IIB result given in Eq. (1.3) of [29], the remnant non-vanishing structure constants take the simple form:

$$d_{q+1} = -\tfrac{1}{4}(q - 3), \; q = 1, 5, \qquad c_{2,6} = \tfrac{1}{2} \quad . \tag{12}$$

The algebra we obtain is in precise agreement with that of the 10d N=1 heterotic supergravity given in Eq. (1.4) of [32]. Let us denote this algebra as $\mathcal{G}_{IB}$. We emphasize that, thus far, we have not included the Yang-Mills gauge sector of the nonmaximal 10d supergravity-Yang-Mills theory.

Let us now address the question of how the global symmetry algebra of the heterotic-type I nonmaximal supergravity relates to the symmetry algebra of M theory. West has provided mounting evidence in favor of the conjecture that the symmetry algebra of M theory is the rank eleven very-extended algebra E_{11} [31], a summary appears in Appendix B. If we compare the generators and commutation rules given above with the Chevalley basis for the algebra $E_8^{(3)}$, written in either its IIA or IIB guise as shown in [31], we find that we are missing some of the positive root generators in either formulation. We have all of the generators, $E_a = K_{a+1}^a$, $a=1, \cdots, 9$, of $SL(10, \mathbf{R})$. In the IIA formulation, given in Eq. (4.4) of [31], we are missing the roots corresponding to the R-R one-form, and NS-NS twoform, namely, $E_{10} = R^{10}$, and $E_{11} = R^{910}$. In the IIB formulation, we are missing the

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roots labelled $E_9 = R_1^{910}$, and $E_{10} = R_2$, arising, respectively, from the NS-NS two-form potential, and R-R scalar. It is clear we cannot build a full $E_8^{(3)}$ algebra from the restricted set of generators in $\mathcal{G}_{IB}$.

In [32], it was pointed out that a different rank eleven very-extended algebra, namely, the very-extension of the D_8 subalgebra of E_8, can be spanned by the generators of $\mathcal{G}_{IB}$. But we should note that such a construction is somewhat unmotivated from the viewpoint of any relationship to the type II theories, to eleven-dimensional supergravity, or to M theory: the authors of [32] make the choice $E_a = K_{a+1}^a$, $a=1, \cdots, 9$, $E_{10} = R^{910}$, and $E_{11} = R^{5678910}$. This choice of simple roots is shown to generate the very-extended algebra $D_8^{(3)}$. Appending a one-form generator to this set converts the $D_8^{(3)}$ algebra to $B_8^{(3)}$ [32]. However, it should be noted that, since the two-form and six-form potentials are Hodge dual to each other in ten dimensions, a construction which includes both in the simple root basis is quite different in spirit from that of the $E_8^{(3)}$ algebras underlying the IIA and IIB theories [31].

We will suggest a different direction towards uncovering an $E_8^{(3)}$ algebraic structure in supergravity theories with sixteen supercharges. Since we already have the requisite two-form potential, respectively, the R-R, or NS-NS, two-form of the type IB, or heterotic, supergravities, our goal will be to identify a one-form potential that can play the role of the positive root generator labelled E_{10} in the IIA formulation of the $E_8^{(3)}$ algebra. A hint is provided by our understanding of the duality web linking the zero slope limits of the circle compactifications of the type I, type IIA, IIB, and heterotic string theories, along with M theory compactified on $S^1 \times S^1/\mathbb{Z}_2$. Upon compactification on a circle, the heterotic string theories acquire an abelian one-form potential, namely, a Kaluza-Klein gauge boson. This perturbative gauge field is known to play a crucial role in six-dimensional weak-strong IIA-heterotic string-string duality: it maps under a weak-strong coupling duality to the R-R one-form potential of the IIA string theory compactified on K3.[10] As the simple root generator labelled E_{11} in the IIA formulation of $E_8^{(3)}$, we choose, respectively, the NS-NS two-form potential of the IIA string or the two-form potential of the heterotic string. Note that these are mapped to each other under string-string duality. As an aside, the reader may wonder why we had to compactify all the way to six dimensions to see these equivalences, but this methodology is in keeping with how the Cremmer-Julia hidden symmetries of the type II theories were discovered. The full global symmetries only become manifest in the dimensionally-reduced supergravity theories, but this can be taken as a hint towards discovering a higher-dimensional correspondence. In summary, identifying the Kaluza-Klein one-form, and the two-form potential, as the two missing positive root generators in the Chevalley basis should plausibly allow one to demonstrate an $E_8^{(3)}$ global symmetry algebra in the circle compactifications of the heterotic supergravities.

The evidence for an $E_8^{(3)}$ global symmetry algebra is even clearer in the case of the circle-compactified type IB supergravity. Upon compactification to nine dimensions, the type I theory

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[10]To be more precise, compactifications of the IIA theory on K3 are described by a $(19, 3)$ cohomology lattice characterizing classical K3 surfaces. It is the quantum extension to a $(20, 4)$ quantum cohomology lattice, as a consequence of introducing a flux for the R-R one-form potential, that completes the isomorphism of the IIA theory compactified on K3 to the heterotic string compactified on T^4. The quantum cohomology lattice is identified with the $(20, 4)$ Lorentzian momentum lattice of the heterotic string [68, 69, 49]. The heterotic theory has 20 abelian one-forms. One of these is distinguished as the partner of the R-R one-form of the IIA theory, and this is the Kaluza-Klein gauge field we have in mind.

can acquire nonabelian gauge symmetries of nonperturbative origin. The type I′ theory has the same p-form gauge potentials as the massive IIA theory, and so the correspondence to the $E_8^{(3)}$ hidden symmetry algebra is particularly transparent. It is important to notice that the supergravity structure of the CHL orbifold is identical with that of the circle compactification. Indeed, if we are correct in our expectation that the circle-compactified theory has an underlying $E_8^{(3)}$ global symmetry algebra, then this will also be true of the CHL orbifold. The distinction between the two theories lies in the Yang-Mills sector: they differ in the rank of the gauge group at generic points in the moduli space, respectively, 17, and 9. Are the precise nonabelian enhanced gauge symmetry groups relevant to this discussion? It has become customary to think not, since it is well-known that the precise nonabelian enhancement varies from point to point in the moduli space. Conventionally, this multiplicity of enhanced symmetry points is expressed in terms of the perturbative T-duality groups of the moduli spaces.

However, if the Kaluza-Klein one-form is absorbed in the nonlinear realization of $E_8^{(3)}$, an especially simple result follows: we find three distinct theories in nine spacetime dimensions with sixteen supercharges. They are characterized by three distinct global symmetry groups: $E_8^{(3)} \times (\mathrm{Spin}(32)/\mathbb{Z}_2)$, $E_8^{(3)} \times (E_8 \times E_8)$, and $E_8^{(3)} \times (E_8)$.[11] The factor in brackets arises from the Yang-Mills sector, the former from the supergravity sector. The strong coupling duals of these heterotic theories are, respectively, the circle compactified type IB, type I′, and type I′, theories in nine spacetime dimensions with corresponding global symmetry groups.

From the perspective of the hidden symmetry algebra, on the other hand, there is hope of finding a tantalizing relationship between supergravity theories having 16 or 32 supercharges: M theory would have a superalgebra given by the fermionic extension of $E_8^{(3)}$ by 32 supersymmetry generators. An alternative extension of the bosonic algebra $E_8^{(3)}$ to a superalgebra with only sixteen supersymmetry generators would simultaneously permit the incorporation of up to 32 Yang-Mills gauge fields. More precisely, we would find that the possible extensions of the global symmetry group are isomorphic to the isolated components of the moduli spaces with sixteen supercharges. In nine dimensions, the only solutions are the rank 16 and rank 8 groups described above. But for moduli spaces in lower dimensions, there is a huge proliferation of isolated components to the moduli space, with the enhanced gauge symmetry varying from point to point, but with identical abelian subgroup at all points in the moduli space. Thus, a precise specification of the global remnant of this abelian subgroup is an accurate characterization of the CHL orbifold. Notice that the nontrivial global remnant of the abelian gauge symmetry simply reflects the action of the orbifold group on the toroidal spacetime. This is a slightly more physical explanation of how physics differs in the CHL moduli spaces.

In summary, let us reformulate our conjecture for M Theory succinctly [1, 2] in light of what we have learned from this brief survey of the hidden symmetry algebras of nine, ten, and eleven, dimensional supergravity theories. Given the striking evidence that the rank eleven very-extension of the Lie algebra E_8 incorporates the full spectrum of NS-NS and R-R pbranes, $-2 \leq p \leq 9$, we will conjecture that the hidden symmetry algebra of supergravity theories with sixteen supercharges

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[11]We are using the same notation for the Yang-Mills gauge group and its global remnant. The E_8 in the case of the CHL orbifold is realized at level two of the worldsheet affine Lie algebra. From the perspective of spacetime, the $\mathbb{Z}_2$ orbifold structure indicates the necessity for care in determining the precise global remnant of the E_8 gauge symmetry.

takes the form $E_8^{(3)} \times G$, where G is the global remnant of an appropriate finite-dimensional Yang-Mills gauge group. We conjecture further that nonperturbative String/M theory is the locally supersymmetric extension of the algebra $E_8^{(3)} \times G$ with sixteen supercharges realized on the field of unitary $N \times N$ matrices. A particular limit of this superalgebra will recover a full 32 supercharges at the cost of making G trivial.

5 Open Questions

A proposal as radical as that described in [1, 2] has few concrete conclusions in comparison with the Pandora's box of fascinating questions it opens up for future investigation. We include the list of the most significant, and accessible, of the questions given in [2] below, along with some issues specific to the framework of nonlinear realizations and Lorentzian Kac-Moody algebras [31, 33, 34, 36, 55]:

- What is the physical significance of the higher level roots of the Lorentzian very-extended Lie algebra? This is the most important issue in extending the arguments *in favor* of E_{11} as the symmetry algebra of M theory into a fully convincing *proof.* All Lorentzian Kac-Moody algebras exhibit the level structure, described briefly in Appendix A, and it is here that E_{11} has consequences for M theory beyond those evident in its low energy supergravity limit. Fortunately, there is considerable ongoing investigation of the level structure, both in the context of the over-extended algebra $E_{10} = E_8^{(2)}$ [36, 33, 55], and in the case of $E_{11} = E_8^{(3)}$ [97, 35, 34, 31]. The general feature common to all of these analyses is to attempt an isomorphism between all of the known classical supergravity solutions, including composite branes and bound states, to the root system of the Lie algebra. There will also be new solutions which do not have a supergravity correspondence; examples in the case of eleven-dimensional supergravity have been given by West in [31]. An interesting point raised by West [31] that deserves further investigation is the possibility of an isomorphism between the process of group multiplication between group elements corresponding to two elementary branes, and the formation of a composite brane or bound state. How should one interpret such states, and what is their correspondence in the matrix theory framework?

- The absence of a clear picture for the origin of spacetime in proposals for nonlinear realizations of the hidden symmetry algebra [36, 33, 34, 24, 31] has made it difficult thus far to explore their relationship to more traditional organizing principles for the pbrane spectrum of type II string/M theory, such as K-theory [78, 79].[12] We should note that the precise role of self-duality in the context of K theory is also unclear. Preliminary steps could be to understand the relationship of K theory to the doubled field formalism of dimensionally-reduced supergravities [57], a precursor to West's nonlinear realization [24, 31], also based on the notion of self-duality [56]. Given the recent detailed understanding provided by West in [31] of the full pbrane spectrum of the type II string theory, M theory, and of eleven-dimensional supergravity, it must surely be possible to make contact with at least some of the results in [79]. Recent work in this direction has exploited the worldsheet correspondence, invoking the framework of RG flows in the larger space of generic two-dimensional field theories, thus incorporating

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[12]I thank Clifford Johnson for raising this question.

unstable branes. Do unstable branes find a natural setting within the framework of nonlinear realizations?

- The elucidation of an $E_8^{(3)}$ algebra with Chevalley basis chosen from among the generators of the global symmetry algebra of the circle-compactified type I-I′-heterotic string theories proposed by us in [2], and reviewed in section 4, needs to be completed. In particular, West has made the interesting observation that the IIA-IIB T-duality transformation simply reflects the bifurcation symmetry of the $E_8^{(3)}$ Dynkin diagram at its central node, interchanging the Dynkin diagrams of its two inequivalent A_9 subalgebras [31]. This argument can clearly be adapted to the T-duality symmetry relating the type I and type I' string theories. Or, to that relating the two circle-compactified heterotic string theories. The details need to be verified.

- As a further check, the conjectured hidden symmetry algebra, $E_8^{(3)} \times E_8$, for the first of the CHL models [43] needs to be verified. In fact, we have conjectured the appearance of an $E_8^{(3)}$ algebra in the supergravity sector of *any* of the vast proliferation of isolated theories with sixteen supercharges [73]. It should be noted that our new perspective on the hidden symmetry algebra takes seriously the strong-weak dualities linking the heterotic, type IB, and type IIA, string theories: a theory of sixteen supercharges is self-dual, and it would be meaningless to have different hidden symmetry algebras pertaining to the different string theories. Thus, while the perturbative T-duality group of the type II theory is, in fact, incorporated in E_{11}, the Lorentzian $O(16+d, d)$, $d < 10$, extension familiar from the toroidally-compactified heterotic string is *not* contained within E_{11}. Our conjecture is that E_{11} is the hidden symmetry algebra of the type I-heterotic supergravity with sixteen supercharges: it is unchanged for *all* of the CHL theories. Of course, the abelian subgroup of the nonabelian gauge symmetry characterizing generic points in a given moduli space, and hence G, will be different for each of the latter CHL theories. It just so happens that in nine dimensions there are no Wilson lines that permit either a breaking, or enhancement, of the E_8 Yang-Mills gauge symmetry.[13]

- Next, we must ask about theories with fewer supercharges: note that the field theory with eight supercharges includes in its moduli space both subspaces, or isolated points, with 4, or 0, supersymmetries, at finite distances in the moduli space. The appearance of matter fields is an added wrinkle. Can the framework of nonlinear realizations be adapted to supergravity theories coupled to chiral matter? This is a beautiful open question that first needs to be addressed purely in a field theoretic setting. Our conjecture is that the theory with eight, or fewer, supercharges has a hidden symmetry algebra that is *smaller* than $E_8^{(3)}$, possibly a subalgebra. The N=3 theories with 12 supercharges, recently revived by Frey and Polchinski [77], offer an interesting half-way point between the 16, 32 supercharge theories and theories with 8 or fewer supercharges.[14] They have well-defined moduli spaces with flat potential, while including examples of three-form flux compactifications that share many features of semi-realistic, N=1 flux compactifications with avenues for moduli stabilization. If a suitable generalization of the framework of nonlinear realizations and hidden symmetries can indeed be

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[13]I thank Arjan Keurentjes for requesting this clarification.
[14]I thank Andrew Frey for stimulating my interest in this subject.

found for such semi-realistic supergravity theories, it should be straightforward to implement within the context of matrix theory.

- Once the precise nature of the hidden symmetry algebra of the type I-I$'$-heterotic theories with sixteen supercharges has been pinned down, and which we have conjectured will take the form $E_8^{(3)} \times G$, what new physics becomes accessible? It is a remarkable fact that there exists a *unique* assignment of phases in the bosonic $E_8^{(3)}$ algebra corresponding to an eleven-dimensional theory with, respectively, Minkowskian (1,10), or Euclidean (0,11) signature [95]. As has been shown by Keurentjes [95], every other self-consistent choice of phases for E_{11} results in a spacetime with two, or more, timelike directions. Furthermore, the Euclidean case also corresponds to a bosonic E_{11} algebra without supersymmetric extension [95], precisely as one would expect given its natural physical interpretation as the symmetry algebra of M theory at finite temperature. This Euclidean symmetry algebra should be of great interest in the context of M theory cosmology, as we now explain.

We will offer a suggestive interpretation for the principle subalgebra of the Euclidean signature bosonic E_{11} algebra. As expected from generic considerations [30], every Lorentzian Kac-Moody algebra has a principal $SO(1,2)$ subalgebra, and it is natural to seek its physical interpretation. Based on our understanding of the String/M duality web in 11, 10, and 9, spacetime dimensions, and given the pivotal role played by the nine-form potential and its (-1)-form dual, it is natural to identify the parameters of the $SO(1,2)$ subalgebra, roughly, as follows.[15] Labelling them as R_0, R_{10}, and R_9, respectively, suggests a natural identification with inverse temperature, β, string coupling, g, and cosmological constant M. The latter is Roman's mass parameter, later interpreted by Polchinski as D8brane charge [38, 10]. Notice that the two-parameter subspace (β, g), whose rough correspondence with the radius of coordinates (X^0, X^{10}) is well-known, is also the classic phase space parameterization relevant for the study of the dynamics of a finite temperature gauge theory [61]. Supplementing this with the cosmological constant gives a natural three-parameter phase space relevant for discussions of cosmology: the thermal dynamics of the Universe, inclusive of gravity [61, 62]. It should be emphasized that the principal $SO(1,2)$ algebra should *not* be confused with the corresponding subalgebra of the spacetime Lorentz algebra.

In recent work on string thermodynamics [61], we have pointed out that there is a fundamental conceptual barrier to proposals for a *microcanonical* description of the string ensemble: perturbative string theory is inherently a background-dependent theory. Thus, we cannot escape the "heat-bath" represented by the spacetime geometry, and additional background fields: *any* self-consistent discussion of string thermodynamics must therefore be relegated to the canonical ensemble. Fortunately, there is a first-principles framework for the canonical ensemble provided by the world-sheet path integral formalism, originally pointed out in [86]. On the other hand, from the perspective of quantum cosmology, the ensemble of interest *is* the microcanonical ensemble of the fundamental degrees of freedom: the Universe is a closed system, and there is no "heat bath" one can point to. The reduced matrix models we have described in this paper offer a self-consistent starting point in which to formulate the mi-

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[15]We should emphasize that this is very rough intuition. Thus, we are not making any clear statement on the precise topology of the group, discrete identifications might be necessitated.

crocanonical ensemble of the fundamental degrees of freedom. This opens up the exciting possibility of a genuinely nonperturbative formulation for black hole thermodynamics and quantum cosmology.

- Next, the detailed understanding of the supersymmetric extension of the $E_8^{(3)} \times G$ hidden symmetry algebra is of profound importance, especially in light of our conjecture for the appearance of the theories with 32 supercharges, and no Yang-Mills sector, as special limits in the superalgebra with sixteen supercharges. Fortunately, in recent work [55], some of the tools necessary for such an analysis have been developed. We should note, however, that the focus of [55] is the rank 10 hyperbolic Kac-Moody algebra E_{10}. Recall that the A_{10} subalgebra of E_{11} has the natural interpretation as originating in the gravity sector of a *noncompact* 11d theory. E_{10} has an A_9 subalgebra, but not A_{10}. Not surprisingly, the framework of [55] also cannot incorporate a space-filling nine-brane, which would have to couple to a rank eleven field strength.

- Given the precise details of the hidden *super*algebra, an analysis along the lines of [19, 21, 22] is necessary to verify whether there exists a manifestly supersymmetric and covariant supergravity and Yang-Mills Lagrangian where this symmetry is manifest? If so, by the arguments in this paper, it is clear that there exists a corresponding zero-dimensional supermatrix Lagrangian where this symmetry is also manifest. We should emphasize that whether one implements our nnperturbative proposal for M theory in a Lagrangian or Hamiltonian framework is, in part, a matter of taste. Each perspective offers distinct advantages. Nevertheless, for the reasons mentioned above, the matrix Lagrangian formulation should remain a priority for future investigations.

- As an aside, we should note that an important issue raised both by our focus on the principal three-parameter subalgebra of $E_8^{(3)}$, and by its realization in a locally supersymmetric unitary matrix model, is the possibility of an undiscovered relation to the famous *supermembrane theory*, a conjectured theory of fundamental supermembranes [75, 76]. Does the locally supersymmetric matrix model represent a regularization of the three parameter manifold of the principal subgroup, analogous to the regularization of the worldvolume of the supermembrane provided by the rigid unitary matrix model [76]? These are open questions that should shed light on the large N continuum limit of the reduced matrix model.

- Finally, coming to the crucial open questions in the matrix model framework, there is the issue of what comes beyond leading order in large N in the matrix model: what is the significance of the off-diagonal elements of the variables in the reduced matrix model? Notice that there is an obvious extension to the notion of the double-scaling limit familiar from the $c = 1$ matrix model: namely, $\lim_{N \to \infty, g \to 0}$, with $g^\alpha N^\beta$ held fixed. The parameters (α, β) take an appropriate range of values for members in the discrete series of the gravitationally-dressed unitary conformal field theories with central charge $c \leq 1$ [4]. The generalization to large N limits with multiple-scaling was pointed out in our earlier works [1]. Since we have a full range of background fields, $(g = e^{\bar{\phi}}, \bar{A}_{c_1}, \bar{C}_{c_1}, \bar{C}_{c_1 c_2}, \cdots, \bar{C}_{c_1 \cdots c_9})$, where $g = (M_{11} R_{10})^{3/2}$, and the single mass scale, M_{11}, there are many possible inequivalent, multiple-scaling limits: a suitable combination of powers of N, M_{11}, and the background fields, can be held fixed, in the limit that we take $N \to \infty$. Here, M_{11} has been taken to be the eleven-dimensional Planck

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mass. The precise powers of M_{11} that enter into taking the large N limit can vary, depending on whether we wish to match to an eleven, or ten-dimensional, continuum field theory. For example, the ten-dimensional string mass scale is related as follows: $m_s = \alpha'^{-1/2} = M_{11}^{3/2} R_{10}^{1/2}$. We should emphasize the fact that it was essential that the matrix Lagrangian framework allow for a wide range of inequivalent large N limits, since it would not otherwise be possible to explain the multitude of known effective dualities relating M theory ground states.[16]

- Perhaps the most important open question is the comparison of corrections to the large N limit of the matrix model, calculated with the specific choice of scaling appropriate for matching to a particular string supergravity, with the higher order in α' corrections to the string spacetime effective Lagrangian.[17] We should remind the reader that the precise form of the low energy spacetime effective Lagrangian for string theory has not been systematically calculated beyond quartic order in the inverse string tension [21, 22], and that too only in the case of the heterotic string. This is unfortunate, given that the techniques for the systematic derivation of these terms from string amplitude calculations, or based on duality symmetries of the effective action, have been known for many years [21, 23, 39]. In the past, this was explained by the absence of any direct physical motivation for a *comprehensive* analysis. For example, it was common to focus on the particular subset of terms that had the potential to mediate some new physics beyond the standard model. But given our current understanding of M theory, it would now seem that there is *strong motivation* for a renewed effort at obtaining a comprehensive analysis of the spacetime effective Lagrangian. We emphasize that it is only at higher orders in the α' expansion that we can successfully test any conjectured nonperturbative proposal for M theory beyond agreement with the supergravity prediction.

In summary, we believe this could be the beginning of an exciting period in the search for a more fundamental description of String/M theory that transcends its weakly-coupled perturbative limits.

ACKNOWLEDGMENTS

Appendix A: The Very-Extension of a Lie Algebra

[16] As an illustration, consider the unusual suggestion for incorporating nonsupersymmetric, metastable ground states such as the heterotic orbifold with *timelike* linear dilaton potential and matter fields with $D > 1$, recently explored in [80]. The relevant scaling limit would requires matching to a matrix model with extended symmetry algebra embedding $SL(10 + n, \mathbf{R}) \times SO(32 + n) \times SO(n)$, where n is the number of extra matter bosons, and with nontrivial background dilaton and three form flux. Note that the generators of $SL(10 + n, \mathbf{R})$ can nevertheless be rescrambled into generators of $E_8^{(3)}$, except that some of them will now appear at higher level in the Lorentzian algebra, as discussed from a rather general viewpoint in [31]. I thank Simeon Hellerman for discussion of string theory backgrounds with timelike dilaton potential.

[17] I would like to thank Sanefumi Moriyama for asking this question.

The introductory material in this appendix can be found in the classic texts [91, 92], as well as the well-known review article [93]. The notion of a very-extended algebra first appears in Gaberdiel, Olive, and West [29]. Our discussion of the very-extension of E_8 is based on section 5 and the appendices of this reference. We urge the reader to consult the original [29] for a far more complete treatment. A readable introduction to some novelties in the representation theory of very-extended algebras, the subject of current research, can be found in section 2 of [35].

As is well-known, Cartan's classification of the classical Lie algebras, the A_n, B_n, C_n, and D_n series, extends to three exceptional cases, namely, E_n, with n=6, 7, and 8 [94]. Every such finite-dimensional semi-simple Lie algebra has an infinite-dimensional affine extension, better known in the physics literature as a current algebra [40], or as an affine Lie algebra. The two-dimensional field theoretic realizations of affine Lie algebras have been the basis of considerable work in rational conformal field theory and string theory [93]. While affine Lie algebras are the best-studied examples of the Kac-Moody algebras, more generally, they are characterized as follows [91, 92]. We can write the generators of any Kac-Moody algebra, $\mathcal{G}$, in what is known as the Chevalley basis: the positive and negative simple root generators, $E_a^{\pm}$, are the generalization of the raising and lowering operators, $J_{\pm}$, of the angular momentum group $SU(2)$, familiar to every quantum physicist. Likewise, the role of the single eigenoperator of $SU(2)$, usually denoted J_3, is more generally played by the Cartan subalgebra of $\mathcal{G}$. This is the maximal subset of mutually-commuting generators, denoted by H_a. The number of generators in the Cartan subalgebra gives the rank, r, of the algebra. We have the familiar commutation relations:

$$[H_a, E_b^{\pm}] = \pm A_{ab} E_b^{\pm}, \quad [E_a^+, E_b^-] = \delta_{ab} H_a \quad . \tag{13}$$

The matrix A_{ab} is known as the Cartan matrix. For the simple Lie algebras in Cartan's classification, its determinant is positive-definite: $\det(A) > 0$. It is important to note that the Cartan matrix uniquely defines a corresponding Kac-Moody algebra: given the entries of A_{ab}, we can use the commutation relations in Eq. (13) and what are known as Serre relations:

$$[E_a^+, \cdots [E_a^+, E_b^+] \cdots] = 0, \quad [E_a^-, \cdots [E_a^-, E_b^-] \cdots] = 0 \quad , \tag{14}$$

to reconstruct the generators and roots of the Kac-Moody algebra. Labelling the simple roots α_a, a=1, $\cdots$, r, where r is the rank of $\mathcal{G}$, the Cartan matrix can be parametrized as follows:

$$A_{ab} = 2\frac{(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)} \quad , \tag{15}$$

from which it follows that A_{aa}=2. The generators can be normalized such that all off-diagonal entries of the Cartan matrix are negative integers, or zero. The entries of the Cartan matrix can be conveniently encoded by an unoriented graph with r nodes and adjacency matrix $2\delta_{ab} - A_{ab}$, known as the Dynkin diagram. It specifies the Cartan matrix uniquely upto a re-labelling of rows and columns. Thus, the simply-laced Dynkin diagrams contain only single links between nodes, since all off-diagonal entries of A_{ab} are either -1, or 0. Non-simply-laced Dynkin diagrams can include multiple links, corresponding to the appearance of other negative integers. A disconnected Dynkin diagram implies that the Cartan matrix takes block-diagonal form, and that the algebra decomposes into simple commuting factors. Notice that the Kac-Moody algebra is invariant under the set of involutions defined by:

$$E_a^+ \to \eta_a E_a^-, \quad E_a^- \to \eta_a E_a^+, \quad H_a \to -H_a \quad , \tag{16}$$

where the $\eta_a = \pm 1$ for every a. We emphasize that each self-consistent choice of phases, η_a, corresponds to a distinct Kac-Moody algebra. This freedom in the choice of phases becomes especially significant when attributing a spacetime interpretation to the related global symmetry algebra [24, 95], as discussed in Appendix B for E_{11}.

Every simple root generator is isomorphic to a vector α_a^i, such that $H_a = 2\frac{\alpha_a^i H_i}{(\alpha_a, \alpha_a)}$, where the H_i define an alternative basis for the Cartan subalgebra known as the Cartan-Weyl basis. Thus, to any root, α, we can associate generators, $E_\alpha^\pm$, such that $[H_i, E_\alpha^\pm] = \pm \alpha^i E_\alpha^\pm$. Notice that each set of three generators, $\{E_\alpha^+, \alpha^i H_i, E_\alpha^-\}$, defines a distinct A_1 subalgebra of $\mathcal{G}$. The r simple root vectors, α_a^i, span an r-dimensional vector space known as the root-lattice of $\mathcal{G}$, denoted $\Lambda_{\mathcal{G}}$. It is convenient to single out the so-called highest root vector in this lattice, usually denoted θ, normalized as $\theta^2 = 2$, and parameterized as follows: $\theta = \sum_{i=1}^r n_i \alpha_i$, where the $n_i = 2\frac{(\theta, \lambda_i)}{(\alpha_i, \alpha_i)}$ are known as the Kac labels. The λ_i are the r distinct fundamental weight representations of the rank r simple Lie algebra, defined by the relations $2\frac{(\lambda_j, \alpha_i)}{(\alpha_i, \alpha_j)} = \delta_{ij}$ [94, 91, 92].

Given any finite dimensional semi-simple Lie algebra $\mathcal{G}$, we can construct its affine extension, $\mathcal{G}^{(1)}$, by the addition of a node to its Dynkin diagram. This construction is reviewed in the classic paper of Goddard and Olive [93].[18] We begin with the unique Lorentzian even self-dual lattice of dimension two, $\Pi^{(1,1)}$, with norm $x \cdot y = x_1 y_1 - x_{-1} y_{-1}$. $\Pi^{(1,1)}$ is conveniently expressed in terms of a light-cone basis, mapping lattice vectors $x, y \to z, w$, where [93, 29]:

$$z = \frac{1}{\sqrt{2}}(x_1 + x_{-1}), \quad \bar{z} = \frac{1}{\sqrt{2}}(x_1 - x_{-1}), \quad \text{and} \quad z, \bar{z}, w, \bar{w} \in (m, n), \quad \forall\, m, n \in \mathbb{Z} \quad . \quad (17)$$

The primitive null vectors of $\Pi^{(1,1)}$ are $k = (1, 0)$ and $\bar{k} = (0, 1)$. Let us append $\Pi^{(1,1)}$ to the root lattice of $\mathcal{G}$, and consider the subspace of vectors in $\Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)}$ that are orthogonal to the primitive null vector k. It is clear that this subspace includes all of the root vectors of $\mathcal{G}$, spanned by the original set of simple roots $\{\alpha_i\}$. The enlarged span of the affine extended algebra is defined by appending an additional simple root, α_{-1}, also referred to as the extended, or affine, root. Thus, the affine extension of $\mathcal{G}$ has rank $r + 1$. The extended root takes the form:

$$\alpha_{-1} = k - \theta, \qquad k \cdot k = 0, \qquad (\alpha_{-1}, \alpha_{-1}) = 2 \quad . \quad (18)$$

Notice that α_{-1} has been written as an $(r+2)$-dimensional vector, reflecting the fact that the root-lattice of $\mathcal{G}^{(1)}$ is an $(r+1)$-dimensional projection from the $(r+2)$-dimensional lattice, $\Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)}$. Clearly, the Cartan matrix of the affine extension of $\mathcal{G}$ will have one additional row, and one additional column, with entries given by the scalar products of α_{-1} with the simple roots of $\mathcal{G}$:

$$A_{i,r+1} = 2\frac{(\alpha_{-1}, \alpha_i)}{(\alpha_i, \alpha_i)}, \qquad A_{r+1,i} = (\alpha_{-1}, \alpha_i) \quad . \quad (19)$$

It is clear that the determinant of the extended Cartan matrix vanishes: $\det A_{ab}(\mathcal{G}^{(1)}) = 0$. As an aside, we comment that $\mathcal{G}^{(1)}$ has been denoted interchangeably by $\mathcal{G}^+$ in the recent literature [93, 29, 31, 32].

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[18]We have simplified the notation in [93] [29] as follows: $\mathcal{G}^{(n)}$ with $n=1$, 2, and 3, will denote, respectively, the extension, over-extension, and very-extension, of the Lie algebra $\mathcal{G}$. This corresponds to successive extensions of the rank r simple root basis $\{\alpha_i, i = 1, \cdots, r\}$, by the addition of simple roots denoted α_{-n}, where $n=1$, 2, and 3.

This naturally suggests that we ask what rank $(r + 2)$ algebra might correspond to the full extension of the root-lattice, $\Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)}$? The mathematically well-defined way to address this question is to first return to the Dynkin diagram of $\mathcal{G}^{(1)}$, adding a single link to the affine node [93]. This defines what is known as the over-extension of $\mathcal{G}$: $\mathcal{G}^{(2)} = \mathcal{G}^{++}$ [93, 29]. The over-extended root takes the form:

$$\alpha_{-2} = -(k + \bar{k}), \qquad k \cdot k = \bar{k} \cdot \bar{k} = 0, \qquad (\alpha_{-2}, \alpha_{-2}) = 2 \quad . \tag{20}$$

The Cartan matrix of $\mathcal{G}^{(2)}$ is extended by the new entries:

$$A_{i,r+2} = 0, A_{r+2,i} = 0, A_{r+1,r+2} = A_{r+2,r+1} = -1. \tag{21}$$

Notice that the determinant of the Cartan matrix is non-singular and negative-definite, since $\mathcal{G}$ was assumed to be a finite dimensional semi-simple Lie algebra:

$$\det A_{\mathcal{G}^{(2)}} = 2 \det A_{\mathcal{G}^{(1)}} - \det A_{\mathcal{G}} = -\det A_{\mathcal{G}} \quad . \tag{22}$$

Such a Kac-Moody algebra is said to be *Lorentzian*. The over-extension of a finite dimensional semi-simple Lie algebra is an especially simple example of a Lorentzian Kac-Moody algebra. We reiterate that, by construction, the root lattice of $\mathcal{G}^{(2)}$ has Lorentzian signature: $\Lambda_{\mathcal{G}^{(2)}} = \Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)}$. The seemingly innocuous extension to a root-lattice with indefinite norm has profound consequences. Notice that the root system of a Lorentzian Kac-Moody algebra includes both real and *imaginary* roots, namely, those with $\beta^2 < 0$ [91, 53, 97, 36]. The representation theory of Lorentzian algebras turns out to be full of surprises: unlike what happens in a finite-dimensional Lie algebras, the adjoint representation is no longer a highest weight representation, nor can it be constructed as the tensor product of fundamentals. In fact, the adjoint representation can contain within it some of the fundamental representations of the algebra! We caution the reader that while the representation theory of the finite-dimensional and affine Lie algebras is known in explicit detail, only partial results are available in the Lorentzian cases. But it is encouraging that the standard tools of Kac-Moody algebras: namely, the characterization of the root system with respect to the Weyl group of reflections, the use of the Weyl-Kac character formula for the computation of root multiplicities, and the well-known Peterson and Freudenthal identities, hold just as well for the Lorentzian cases [91, 92]. The explicit details of the representation theory of the over and very extended algebras are currently under investigation by a number of groups [53, 29, 97, 36].

A further extension of the algebra is enabled by the addition of a link to the over-extended node. This defines what is known as the very-extension of $\mathcal{G}$: $\mathcal{G}^{(3)} = \mathcal{G}^{+++}$, introduced in the work of Olive, Gaberdiel, and West [29]. The root-system of the very-extension is given by the projected subspace of vectors x that are orthogonal to the primitive timelike vector belonging to an additional $\Pi^{(1,1)}$ factor:

$$x \in \Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)} \oplus \Pi'^{(1,1)}, \qquad x \cdot (l - \bar{l}) = 0, \qquad l, \bar{l} \in \Pi'^{(1,1)} \quad , \tag{23}$$

where $l, \bar{l}$ are the primitive null vectors of $\Pi'^{(1,1)}$. This root-system is defined by the addition of the so-called very-extended simple root:

$$\alpha_{-3} = k - (l + \bar{l}), \qquad (\alpha_{-3}, \alpha_{-3}) = 2, \ (\alpha_{-3}, \alpha_{-2}) = -1, \ (\alpha_{-3}, \alpha_{-1}) = (\alpha_{-3}, \alpha_i) = 0 \quad . \tag{24}$$

Recall that k is a primitive null vector of $\Pi^{(1,1)}$. It is easy to see that the Cartan matrix of the very extension simply corresponds to the addition of a row, and column, with mostly zeroes, plus the

single nonvanishing off-diagonal entries, $A_{r+2,r+3}=A_{r+3,r+2}=-1$. The determinant of the Cartan matrix is, once again, negative-definite: $\det A_{\mathcal{G}^{(3)}}=-2\det A_{\mathcal{G}}$.

The weight system of the very-extended algebra can be inferred by tracing its progression thru the iterative construction described above [29]. The result is easy to motivate. In terms of the fundamental weights of the finite dimensional Lie algebra, λ_i, $i=1, \cdots, r$, we have:

$$\lambda_i = \quad \lambda_i - (\lambda_i, \theta)[k - \bar{k} - \frac{1}{2}(l + \bar{l})]$$

$$\lambda_{-1} = \quad -[k - \bar{k} - \frac{1}{2}(l + \bar{l})], \quad \lambda_{-2} = -k, \quad \lambda_{-3} = -\frac{1}{2}(l + \bar{l}) \quad . \tag{25}$$

The weights of the over-extended algebra can be recovered from these expressions by simply setting $l=\bar{l}=0$. For a simply-laced finite-dimensional Lie algebra with simply-laced root-lattice, and dual weight-lattice [94], it is easy to write down the weight-lattice of the over- and very- extensions [29]. The weight-lattice of the over-extension $\mathcal{G}^{(2)}$ is given by:

$$[\Lambda_{\mathcal{G}^{(2)}}]^* = \Lambda_{\mathcal{G}}^* \oplus \Pi^{(1,1)}, \qquad \rightarrow \quad \frac{[\Lambda_{\mathcal{G}^{(2)}}]^*}{\Lambda_{\mathcal{G}^{(2)}}} = \mathbb{Z}_{\mathcal{G}} \quad . \tag{26}$$

Likewise, for the very-extended algebra, $\mathcal{G}^{(3)}$, the root-lattice and weight-lattice, respectively, take the form:

$$\Lambda_{\mathcal{G}^{(3)}} = \Lambda_{\mathcal{G}} \oplus \Pi^{(1,1)} \oplus \{(m, -m) : m \in \mathbb{Z}\}, \quad [\Lambda_{\mathcal{G}^{(3)}}]^* = \Lambda_{\mathcal{G}}^* \oplus \Pi^{(1,1)} \oplus \{(n, -n) : 2n \in \mathbb{Z}\} \quad . \tag{27}$$

Notice that the roots and weights of the rank $(r + 3)$ algebra are expressed here as vectors in an $(r + 4)$-dimensional vector space. We can infer that their coset takes the form:

$$\frac{[\Lambda_{\mathcal{G}^{(3)}}]^*}{\Lambda_{\mathcal{G}^{(3)}}} = \mathbb{Z}_{\mathcal{G}} \times \mathbb{Z}_2 \quad . \tag{28}$$

A more pedestrian approach to the representation theory of very extended algebras that eschews the traditional, and more abstract, methodology of the Weyl-Kac character formula and Freudenthal identity, can be found in [29, 97, 35, 98]. It has become conventional to label the Dynkin diagram of the very extended algebra as follows: the very, over, and affine, nodes are labelled 1, 2, and 3, starting from left to right along the horizontal, and then continuing from right to left with any nodes above the horizontal. Notice that deletion of a single node of the Dynkin diagram of the very extended algebra, also called the central node, always gives the Dynkin diagram of a finite-dimensional Lie algebra. In the case of $E_8^{(3)}$, the central node is labelled 11, and its deletion gives the Dynkin diagram of its A_{10} gravity subalgebra. Thus, any generic root, β, has a simple root decomposition that takes the form:

$$\beta = n_c \alpha_c + \sum_i n_i \alpha_i \quad , \tag{29}$$

where the α_i are the simple roots of the finite-dimensional algebra following deletion of the central node. The integer n_c is defined as the *level* of the Lorentzian Kac-Moody algebra [29, 53, 97, 35, 98]. Roots at level zero are simply those of the corresponding finite-dimensional Lie subalgebra. It should be noted that the commutators of the algebra preserve the level.

Very-Extension of E_8: The cases of interest in this paper are the affine-, over-, and very- extensions of the simply-laced Lie algebra E_8, with its famous rank eight Euclidean even self-dual root-lattice [94, 39]. In arriving at the Dynkin diagram of $E_{11} = E_8^{(3)}$, we first construct the affine extension of E_8, which is known as E_9, followed by its over-extension, E_{10}, of rank ten. This is the highest rank example of the hyperbolic Kac-Moody algebras. They have been exhaustively classified in the mathematics literature [50, 36]. Λ_{E_8} is spanned by the following eight simple root vectors [94]:

$$
\begin{array}{ll}
\alpha_1 = & (1, +1, 0, 0, 0, 0, 0, 0) \quad \alpha_2 = (1, -1, 0, 0, 0, 0, 0, 0) \\
\alpha_3 = & (0, 1, -1, 0, 0, 0, 0, 0) \quad \alpha_4 = (0, 0, 1, -1, 0, 0, 0, 0) \\
\alpha_5 = & (0, 0, 0, 1, -1, 0, 0, 0) \quad \alpha_6 = (0, 0, 0, 0, 1, -1, 0, 0) \\
\alpha_7 = & (0, 0, 0, 0, 0, 1, -1, 0) \quad \alpha_8 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})
\end{array} \quad . \tag{30}
$$

Appending the extended root α_{-1} determines the affine extension, E_9, where we substitute for the highest root, θ, with the result:

$$
\alpha_{-1} = k - \theta = ((0, 0, 0, 0, 0, 0, -1, 1), (1, 0)) \quad . \tag{31}
$$

The over-extension, E_{10}, is specified by including the additional simple root:

$$
\alpha_{-2} = -(k + \bar{k}) = ((0, 0, 0, 0, 0, 0, 0, 0), (-1, 1)) \quad . \tag{32}
$$

The rank-ten root-lattice of E_{10}, namely, $\Lambda_{E_8} \oplus \Pi^{(1,1)}$, is even, and self-dual, by construction. It coincides with the unique even Lorentzian self-dual lattice of dimension ten, usually denoted $\Pi^{(9,1)}$. Finally, for the very-extension, $E_{11} = E_8^{(3)}$, we include the additional simple root:

$$
\alpha_{-3} = k - (l + \bar{l}) = ((0, 0, 0, 0, 0, 0, 0, 0), (1, 0), (-1, 1)) \quad . \tag{33}
$$

The root system of E_{11} is the projected subspace orthogonal to the primitive timelike vector, $l + \bar{l}$, in the unique even Lorentzian self-dual lattice in dimensions $(10, 2)$:

$$
x \in \Pi^{(10,2)} = \Lambda_{E_8} \oplus \Pi^{(1,1)} \oplus \Pi^{(1,1)}, \qquad x \cdot (l - \bar{l}) = 0 \quad . \tag{34}
$$

As a consequence of the self-duality property of the E_8 lattice, we now have the elegant result:

$$
\frac{[\Lambda_{E_{11}}]^*}{\Lambda_{E_{11}}} = Z_2 \quad . \tag{35}
$$

In summary, notice that, by construction, the rank-eleven algebra E_{11} contains the full Cremmer-Julia E_{11-n}, $n = 0, \cdots, 11$, sequence of hidden symmetry algebras as proper subalgebras. This was, in fact, the original motivation for West's construction [24]. But it is a beautiful consequence that $E_8^{(3)}$ also incorporates the crucial nine-form potential: the generator associated with the very-extended node of its Dynkin diagram, thus unifying Roman's massive type IIA supergravity with both M theory, as well as the massless IIA and IIB supergravities and their toroidal compactifications to lower spacetime dimensions. This is precisely as was required by our conventional understanding of string/M dualities [46, 47, 48, 10, 39].

Appendix B: The Nonlinear Realization of the $E_{11} = E_8^{(3)}$ Algebra

The review paper by Cremmer, Julia, Lu, and Pope [15] contains a detailed explanation of the method of nonlinear realizations from first principles in section 4, and we urge the non-specialist to consult this reference prior to reading the recent work on very-extensions [25, 24, 27, 28, 29, 31, 32, 35]. For completeness, we begin with a brief overview of the straightforward nonlinear realization of the scalar Lagrangian in dimensions $D \geq 6$ in this appendix. The more complicated analysis for the cases $3 \leq D \leq 5$ can be found in the references [14, 15, 26].

We begin with the Lagrangian of the fully-undualized toroidally-compactified eleven-dimensional supergravity in D dimensions, using the notation in [15]. Let us denote the set of dilaton vectors as $\mathcal{A}^i_{[0]j} = b_{ij}$, and the axions collectively as $\mathcal{A}_{[0]ijk} = a_{ijk}$. The i, j, k are internal indices taking values from $1, \cdots, 11-D$; at this stage, one need not distinguish them from tangent space indices. The key observation, dating back to [14], is that in each case the scalars are in one-to-one correspondence with the positive root vectors of the E_{11-n} algebra:

$$
\begin{aligned}
\mathcal{L} = \quad & eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{48}ee^{\mathbf{a}\cdot\phi}F^2_{[4]} - \frac{1}{12}e\sum_i e^{\mathbf{a_i}\cdot\phi}(F_{[3]i})^2 - \frac{1}{4}e\sum_{i<j} e^{\mathbf{a_{ij}}\cdot\phi}(F_{[2]ij})^2 \\
& -\frac{1}{4}e\sum_i e^{\mathbf{b_i}\cdot\phi}(F^i_{[2]})^2 - \frac{1}{2}\sum_{i<j<k} e^{\mathbf{a_{ijk}}\cdot\phi}(F_{[1]ijk})^2 - \frac{1}{2}e\sum_{i<j} e^{\mathbf{b_{ij}}\cdot\phi}(\mathcal{F}^i_{[1]j})^2 + \mathcal{L}_{\mathbf{C-S}} \quad ,
\end{aligned}
\tag{36}
$$

where $\mathcal{L}_{\text{CS}}$ is the dimensional reduction of the $F_{[4]}\wedge F_{[4]}\wedge A_{[3]}$ in eleven dimensions. The notation distinguishes the 1-form field strengths by their origin in the metric, $\mathcal{F}^i_{[1]j}$, or in the three-form potential, $F_{[1]ijk}$, of eleven-dimensional supergravity:

$$
F_{[1]ijk} = dA_{[0]ijk} - \mathcal{A} \wedge d\mathcal{A} \text{ terms}, \quad \mathcal{F}^i_{[1]j} = d\mathcal{A}^i_{[0]j} - \mathcal{A} \wedge d\mathcal{A} \text{ terms} \quad .
\tag{37}
$$

We choose $b_{i,i+1}$ and a_{123} as simple roots; removing a_{123} gives the simple roots of $SL(11 - D, \mathbf{R})$. As is usual, the root-lattice is generated by linear combinations of the simple roots with positive-definite integer coefficients. The Dynkin diagram is as shown in Figure 1. Note that inclusion of the axions from dualized field-strengths in $3 \leq D \leq 5$, namely, $(\mathbf{a}, \mathbf{a}_i, \mathbf{b}_i, \mathbf{a}_{ij})$, fill out the root-lattices of E_6, E_7, and E_8. It is helpful to introduce the following parameterization of roots [26, 15]:

$$
\mathbf{a} = -\mathbf{g}, \quad \mathbf{b}_i = -\mathbf{f}_i, \quad \mathbf{a}_i = \mathbf{f}_i - \mathbf{g}, \quad \mathbf{b}_{ij} = \mathbf{f}_j - \mathbf{f}_i, \quad \mathbf{a}_{ij} = \mathbf{f}_i + \mathbf{f}_j - \mathbf{g}, \quad \mathbf{a}_{ijk} = \mathbf{f}_i + \mathbf{f}_j + \mathbf{f}_k - \mathbf{g} \quad , \tag{38}
$$

where $\mathbf{f}_i$ and $\mathbf{g}$ are $(11 - D)$-dimensional vectors satisfying the relations: $\mathbf{g}\cdot\mathbf{g} = 2\frac{11-D}{D-2}$, $\mathbf{g}\cdot\mathbf{f}_i = \frac{6}{D-2}$, and $\mathbf{f}_i \cdot \mathbf{f}_j = 2\delta_{ij} + \frac{2}{D-2}$. Also, $\sum \mathbf{f}_i = 3\mathbf{g}$. It follows that $\mathbf{b}_{ik} = \mathbf{b}_{ij} + \mathbf{b}_{jk}$, and $\mathbf{a}_{ijk} + \mathbf{b}_{il} = \mathbf{a}_{ljk}$. Identifying the positive roots $\mathbf{b}_{ij}$ and $\mathbf{a}_{ijk}$ with generators E^i_j and E^{ijk}, respectively, we see that they satisfy the commutation relations:

$$
[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k, \quad [E^m_l, E^{ijk}] = -3\delta^{[i}_l E^{|m|jk]}, \quad [E^{ijk}, E^{lmn}] = 0 \quad . \tag{39}
$$

The first two relations are an expression of $SL(11 - D, \mathbf{R})$ covariance. In dimensions $D\geq 6$, the generators E^{ijk} commute. For $D\leq 5$, whether or not they commute depends upon dualizations. Finally, if we include the hidden subgroup $\mathbf{R}_s$, writing the Cartan generators as a vector $\mathbf{H}$, we have:

$$
[\mathbf{H}, E^j_i] = \mathbf{b}_{ij}E^j_i, \quad [\mathbf{H}, E^{ijk}] = \mathbf{a}_{ijk}E^{ijk} \quad \text{no sum} \quad . \tag{40}
$$

Introducing the non-linear realization [14, 15]:

$$
\mathcal{V} = e^{\frac{1}{2}\phi\cdot\mathbf{H}}e^{\mathbf{b}_{ij}E^j_i}e^{\sum_{i<j<k}\mathbf{a}_{ijk}E^{ijk}} \quad , \tag{41}
$$

it can be verified that:

$$dV V^{-1} = \frac{1}{2} d\phi \cdot \mathbf{H} + \sum_{i<j} e^{\frac{1}{2}\mathbf{b}_{ij} \cdot \phi} \mathcal{F}^i_{[1]j} E^j_i + \sum_{i<j<k} e^{\frac{1}{2}\mathbf{a}_{ijk} \cdot \phi} F_{[1]ijk} E^{ijk} \quad , \tag{42}$$

and the entire scalar Lagrangian is expressible in the form:

$$\mathcal{L} = \frac{1}{4} tr \left(\partial \mathcal{M}^{-1} \partial \mathcal{M} \right) \quad , \tag{43}$$

where we define $\mathcal{M} = \mathcal{V}^T \mathcal{V}$, and where the superscript denotes the transpose. Having written the scalar Lagrangian in Meurer-Cartan form, it is helpful to identify the remnant local gauge symmetry. The transformation:

$$\mathcal{M} \to \mathcal{M}' = U^T \mathcal{M} U \quad , \tag{44}$$

where U is a constant element in the global symmetry group, is found to leave the Lagrangian invariant. Thus, the Lagrangian is made out of $K(E_{11-D})$ invariants, where $K(G)$ is the maximal compact subgroup of G, and we have the coset structure $G/K(G)$ for $D \geq 6$.

West's nonlinear realizations of the hidden symmetry algebras underlying the different supergravity theories and M theory is similar in spirit, but brings in many new features [25, 24, 31]. Earlier in the text, a Hodge dual was introduced for each p-form gauge potential, other than the self-dual potentials, in addition to the generators of $GL(n, \mathrm{R})$, the translations, P^a, and scalars, R_s, if present in the supergravity theory. In the case of M theory, with its eleven-dimensional supergravity field theoretic limit, West proposes the following realization of an $E_{11} = E_8^{(3)}$ algebra:

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b, \qquad [K^a_b, R^{c_1 \cdots c_p}] = \delta^{c_1}_b R^{ac_2 \cdots c_p} + \cdots, \qquad p = 3, 6 \quad , \tag{45}$$

along with the additional commutator:

$$[K^a_b, R^{c_1 \cdots c_8, d}] = \left(\delta^{c_1}_b R^{ac_2 \cdots c_8, d} + \cdots \right) + \delta^d_b R^{c_1 \cdots c_8, b} \quad , \tag{46}$$

The K^a_b are the generators of the A_{10} subalgebra of $E_8^{(3)}$. Notice that this only represents the volume-preserving subgroup, $SL(11, \mathbf{R})$, of the expected $GL(11, \mathbf{R})$, and, in addition, that the momentum generators of eleven-dimensional supergravity are not part of the $E_8^{(3)}$. Thus, from this algebraic perspective, if E_{11} is indeed the symmetry algebra of M theory, it does not appear to be an inherently eleven-dimensional theory.[19]

The $+ \cdots$ in the commutation relations denotes the antisymmetrization of all indices. Notice that the nine-index generator is antisymmetrized in only the first 8 indices. There is, of course, no nine-form potential in eleven-dimensional supergravity. The inclusion of a nine-index generator in the purported symmetry algebra is, therefore, a definitive statement that M theory is more than its low-energy limit. A more physical motivation is Roman's IIA mass parameter: inclusion of the nine-index generator in the symmetry algebra of M theory enables a simple relationship with the

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[19]We are describing the most recent formulation given in [31]. We should warn the reader that there has been a significant shift in perspective from West's earliest proposal regarding E_{11}, namely, in [25], to the more recent ideas summarized in [29, 31]. West remarks in [31] that a better name for this conjectured high-energy completion of eleven-dimensional supergravity might be *E-theory*.

hidden symmetry algebra of the massive type IIA supergravity described in the introduction. In addition, note that E_{11} contains as proper subalgebras the entire Cremmer-Julia E_{11-n} sequence. As an aside, the subalgebra generated by $R^{c_1 c_2 c_3}$ and $R^{c_1 \cdots c_6}$ alone was previously identified as the global symmetry algebra of the M5brane in [100], an early motivation for this algebraic approach to M theory. Thus, we have:

$$[R^{c_1 \cdots c_3}, R^{c_4 \cdots c_6}] = 2R^{c_1 \cdots c_6}, \quad [R^{c_1 \cdots c_6}, R^{b_1 \cdots b_3}] = 3R^{c_1 \cdots c_6 [b_1 b_2, b_3]}, \quad [R^{c_1 \cdots c_6}, R^{b_1 \cdots b_6}] = 0 \quad . \tag{47}$$

The Chevalley generators corresponding to positive simple roots of $E_8^{(3)}$ can be identified as follows [31]:

$$E_a = K_{a+1}^a, \quad a = 1, \cdots, 10, \quad E_{11} = R^{91011} \quad , \tag{48}$$

and the rank eleven Cartan subalgebra is generated by:

$$H_a = K_a^a - K_{a+1}^{a+1}, \ a = 1, \cdots, 10, \quad H_{11} = -\frac{1}{3}(K_1^1 + \cdots + K_8^8) + \frac{2}{3}(K_9^9 + K_{10}^{10} + K_{11}^{11}) \quad . \tag{49}$$

Notice that the six-form and nine-index generator do not enter at this level (zero) of the Kac-Moody algebra. However, since the commutator of R^{91011} with generic K_b^a generates all of the remaining components of the three-form potential, the commutator of the three-form with itself generates all components of the six-form and, finally, the commutators of the six-form and three-form yield the components of the nine-index generator, we can span the root-system of $E_8^{(3)}$ with this simple choice of basis. Thus, any generic root in the root-system will be isomorphic to a string of commutators of Chevalley generators, mapping to a unique group element via the nonlinear realization to which we now turn. The non-linear realization of $E_8^{(3)}$ is built up from group elements that take the form:

$$g = \exp\left[\sum_{a \leq b} h_b^a K_a^b\right] \exp\left[\frac{1}{3!} A_{c_1 c_2 c_3} R^{c_1 c_2 c_3}\right] \exp\left[\frac{1}{6!} A_{c_1 \cdots c_6} R^{c_1 \cdots c_6}\right] \exp\left[\frac{1}{8!} h_{c_1 \cdots c_8, d} R^{c_1 \cdots c_8, d}\right] \quad , \tag{50}$$

$A_{[3]}$, and $A_{[6]}$, are to be identified, respectively, with the three-form potential of supergravity, and its Hodge dual. h_b^a is related to the vielbein as shown below, and $h_{c_1 \cdots c_8, d}$ plays the role of a dual field of gravity. Thus, unlike previous proposals such as the doubled-field formalism [15], the notion of duality, and the framework of non-linear realizations, has been extended to the *full* bosonic sector, inclusive of the graviton! West's motivation for a dual-field formalism for gravity comes from an older work by Borisov and Ogievetsky [99], and is described in [25, 24]. Notice that, since the generators of spacetime translations were absent from the $E_8^{(3)}$ algebra, they also do not appear in the group element. This is unlike the proposed nonlinear realizations of the ten-dimensional type II supergravities written down in [25, 27, 28], which are based on the $\mathcal{G}_{\mathrm{II}}$ algebras described in the introduction, and which explicitly include translations. Although it is possible to invoke the nonlinear realization of $E_8^{(3)}$ to develop an isomorphism of group elements to specific eleven-dimensional line elements because of the appearance of the vielbein, as will be illustrated below, the emergence of the translation generators from this algebraic framework remains an interesting puzzle [31, 35]. It is discussed further in section 4.

Let us now discuss the significance of the choice of phases $\eta_a = \pm 1$ in Eq. (32), without which the bosonic $E_8^{(3)}$ algebra has not been unambiguously specified. As shown by West, taking $\eta_1 = +1$, and all remaining η_a negative [24], leads to a hidden symmetry algebra with both an appropriate

supersymmetric extension and a spacetime with Minkowskian signature and a single time direction, $(-,+,\cdots,+)$. Taking all of the η_a negative gives, instead, a spacetime of Euclidean signature, defining what is known as the Cartan involution-invariant subalgebra [24]. This is also a physically well-motivated choice, of obvious relevance to future discussions of string/M theory at finite temperature [62]. Note that West interchangeably invokes either choice of phase in later work [31], since it is clear that the two alternatives simply correspond to a Wick rotation in spacetime.

The fact that every other choice of phases in the bosonic E_{11} algebra leads to an indefinite spacetime metric with two, or more, timelike directions, some of which do not even admit supersymmetric extension, was clarified in a recent paper by Keurentjes [95]. Restricting to E_{11} algebras that also admit a supersymmetric extension, one finds new self-consistent choices of phase correspond to indefinite spacetimes of signature (2,9), (5,6), (6,5), or (9,2) [95]. In other words, the relationship between the supersymmetric extension of an E_{11} algebra and M theory is unique upto freedom in the signature of spacetime. Interestingly, these signatures can be identified with the conjectured M^* and M' eleven-dimensional theories of Hull [96], whose existence was inferred by the application of timelike dualities on the standard (1,10) signature spacetime metric for eleven-dimensional supergravity. This clarifies the important fact that Hull's new solutions [96] do not correspond to distinct eleven-dimensional theories: they belong in a theory with identical hidden symmetries, apart from the different spacetime signature.[20] It is a most satisfying result following from Keurentjes' analysis that invoking either Euclidean (0,11) or Minkowskian (1,10) spacetime signature— each of which has a clear-cut physical interpretation, *uniquely* determines a bosonic E_{11} algebra with unambiguous phase choice. Note also that while the Minkowskian case permits supersymmetric extension, the Euclidean algebra does not, precisely as required by its physical interpretation as the symmetry algebra of finite temperature M theory.

We will conclude this appendix by illustrating: (i) the isomorphism between a specific group element, g, and the line-element of a well-known classical background of supergravity. (ii) the origin of the D8brane of the massive IIA supergravity, and the space-filling D9brane of the IIB supergravity, in specific group elements of $E_8^{(3)}$. We urge the reader to consult [31] for many more examples of such isomorphisms. The field h_b^a is related to the vielbein as follows:

$$e_a^\mu = e^{h_b^b}(e^{\bar{h}})_\mu^a , \qquad \text{where } \bar{h}_b^a = h_b^a - \delta_b^a h_c^c , \tag{51}$$

We start with our favourite line element, for example, the M2brane metric discovered by Duff and Stelle [101]:

$$ds^2 = N_2^{-2/3}(-(dx_1)^2 + (dx_2)^2 + (dx_3)^2) + N_2^{1/3}((dx_4)^2 + \cdots (dx_{11})^2) , \tag{52}$$

with four-form field strength, $F_{1234} = \partial_m N_2^{-1}$, where m labels directions 4, $\cdots$, 11, orthogonal to the worldvolume of the M2brane, and the harmonic function $N_2 = 1 + k/r^2$, and $r^2 = \sum_{m=4}^{11}(x_m)^2$. Thus, we can identify:

$$(e^h)_1^1 = (e^h)_2^2 = (e^h)_3^3 = N_2^{-1/3}, \quad (e^h)_4^4 = \cdots = (e^h)_{11}^{11} = N_2^{-1/6}, \quad A_{123} = N_2^{-1} - 1 , \tag{53}$$

---

[20]Of course, it is not clear at the present time that any physical significance should be attributed to these alternative solutions, in which case the corresponding E_{11} algebras can eventually be discarded.

where it should be noted that the gauge field is specified with respect to tangent space. Substituting into Eq. (54), we construct the corresponding group element:

$$g_{\text{M2}} = \exp\left[-\tfrac{1}{2}\ln N_2\left(\tfrac{2}{3}(K_1^1 + K_2^2 + K_3^3) - \tfrac{1}{3}(K_4^4 + \cdots K_{11}^{11})\right)\right] \exp\left[(1 - N_2)R^{123}\right] \quad , \qquad (54)$$

As explained in [31], such isomorphisms extend to a vast spectrum of classical supergravity backgrounds, including bound states of multiple branes. Based on a large number of examples [31], West argues that the group element corresponding to a specific half-BPS solution of supergravity parameterized by a harmonic function N can always be written in the form:

$$g = \exp\left[-\tfrac{1}{2}\ln N\beta \cdot H\right] \exp\left[(1 - N)E_\beta\right] \quad , \qquad (55)$$

where β is the corresponding root, and E_β the corresponding generator in $E_8^{(3)}$. This is a most remarkable identification.

Perhaps even more striking from our perspective is the detailed correspondence developed in [31] between the generators of $E_8^{(3)}$, and those of the global symmetry algebras of the maximal ten-dimensional supergravities, namely, $\mathcal{G}_{\text{mIIA}}$ and $\mathcal{G}_{\text{IIB}}$. West begins by observing that the Dynkin diagram of $E_8^{(3)}$ has precisely two inequivalent A_9 subalgebras. These are distinguished with respect to the bifurcation at the very-extended node, which can be labelled 8 on the Dynkin diagram of either A_9. Decomposing $E_8^{(3)}$ with respect to the A_9 subalgebra corresponding to the IIA theory, we identify the following Chevalley basis:

$$E_a = K_{a+1}^a, \quad a = 1, \cdots, 9, \quad E_{10} = R^{10}, \quad E_{11} = R^{910} \quad , \qquad (56)$$

and corresponding Cartan subalgebra:

$$\begin{array}{ll}
H_a = & K_a^a - K_{a+1}^{a+1}, \quad a = 1, \cdots, 9 \\
H_{10} = & -\tfrac{1}{8}(K_1^1 + \cdots + K_9^9) + \tfrac{1}{8}K_{10}^{10} - \tfrac{3}{2}R \\
H_{11} = & -\tfrac{1}{4}(K_1^1 + \cdots + K_8^8) + \tfrac{3}{4}(K_9^9 + K_{10}^{10}) + R
\end{array} \quad . \qquad (57)$$

Notice that this only preserves an $SL(10, \mathbf{R})$ subgroup as appropriate for a ten-dimensional theory. R is the IIA dilaton, and R^{10} and R^{910} are, respectively, components of the IIA R-R one-form and NS-NS two-form generators. By inspection of the commutation relations among the K_b^a and p-form generators given in Eq. (2), and Eq. (4), it is easy to see that this choice of basis suffices to generate the full set of p-form generators entering the mIIA global symmetry algebra. Thus, if we exclude spacetime translations, preserving only the $SL(10, \mathbf{R})$ subalgebra of $GL(10, \mathbf{R})$, we find a clear-cut isomorphism between the generators of $E_8^{(3)}$, and a restricted subset of the generators of $\mathcal{G}_{\text{mIIA}}$: $\{(K_b^a, a, b = 1, \cdots, 10), R, (R^{c_1 \cdots c_q}, q = 1, \cdots, 9)\}$. As argued by us in the main text, the symmetry algebra of significance to Matrix Theory is precisely this restriction of $\mathcal{G}_{\text{mIIA}}$.

Clearly, it is therefore possible to identify a specific $E_8^{(3)}$ group element corresponding to each member in the spectrum of $(p + 1)$-form generators in $\mathcal{G}_{\text{mIIA}}$. But, most remarkably, upon substitution in Eq. (55), West succeeds in deducing the line element for the corresponding p-brane in full agreement with previous results [31]. We begin by noting that the p-brane must couple to a $(p + 1)$-form potential with corresponding generator in the $E_8^{(3)}$ algebra. We begin with the

non-linear realization [31]:

$$g_{\text{mIIA}} = \exp\left[\sum_a h_a^a K_a^a\right] \exp\left[\sum_{a<b} h_b^a K_b^a\right] \prod_{q=9}^5 \exp\left[\tfrac{1}{q!} A_{c_1\cdots c_q} R^{c_1\cdots c_q}\right] \prod_{q=3}^1 \exp\left[\tfrac{1}{q!} A_{c_1\cdots c_q} R^{c_1\cdots c_q}\right] \exp\left[AR\right] \quad,$$

(58)

where special attention must be paid to the reverse ordering in the products, with the zero-form acting first and the nine-form acting last. Suppose we wish to deduce the line-element of the supergravity p-brane coupling to the $(p+1)$-form field with corresponding root, β_{p+1}, and corresponding lowest weight generator: $E_{\beta_{p+1}} = R^{1\cdots p+1}$. By inspection of the commutation relations, we can identify this generator as a string of commutators starting with the positive and negative simple roots in the Chevalley basis. This identifies the corresponding $E_8^{(3)}$ root [31]:

$$\beta_{p+1} \cdot H = \left\{\tfrac{1}{8}(7-p)\left(K_1^1 + \cdots + K_{p+1}^{p+1}\right) - \tfrac{1}{8}(p+1)\left(K_{p+2}^{p+2} + \cdots + K_{10}^{10}\right)\right\} + b_p R \quad,$$

(59)

where $b_7 = 0$, and $b_p = \tfrac{1}{2}\eta(p-3)$, for $p \le 6$, and with $\eta=\pm 1$ for NS-NS, R-R, respectively. Substituting in Eq. (55), we have the result:

$$g_p = \exp\left[-\tfrac{1}{2}\ln N \{\cdots\} - \tfrac{1}{2}b_p \ln N_p R\right] \exp\left[(1 - N_p)E_{\beta_{p+1}}\right] \quad,$$

(60)

where $\{\cdots\}$ represents the linear combination of generators appearing within curly brackets on the R.H.S. of Eq. (59). Using the identity:

$$\exp\left[-\tfrac{1}{2}b_p \ln N_p\right] \exp\left[(1 - N_p)E_{\beta_{p+1}}\right] = \exp\left[N_p^{-\tfrac{1}{2}b_p c_{p+1}}(1 - N_p)E_{\beta_{p+1}}\right] \exp\left[-\tfrac{1}{2}b_p \ln N_p R\right] \quad, \quad (61)$$

where the $c_{p+1}=\tfrac{1}{4}\eta(p-3)$ dependence arises from the $[R, R^{p+1}]$ commutator. We can read off the solution for the dilaton and $(p+1)$-form potential in terms of the harmonic function N_p:

$$e^A = (N_p)^{-\tfrac{1}{2}b_p}, \qquad A_{1\cdots p+1} = N_p^{-1} - 1 \quad.$$

(62)

The corresponding line elements take the form:

$$ds^2 = N_p^{-\tfrac{1}{8}(7-p)}(-(dx_1)^2 + (dx_2)^2 + \cdots + (dx_{p+1})^2) + N_p^{\tfrac{1}{8}(p+1)}((dx_{p+2})^2 + \cdots (dx_{10})^2) \quad.$$

(63)

Thus, we recover the well-known results for the line elements of the half BPS pbranes of type IIA supergravity [105, 104]: here, p=1, 5 parameterize the fundamental string and NS5brane, while p=0,2,4,6 parameterize Dpbranes in the R-R sector. The solution with p=7 is ordinary Minkowskian spacetime.

Moving on to the IIB theory with global symmetry algebra $\mathcal{G}_{\text{IIB}}$ described in Eqs. (6), (7), and (8), we make a corresponding identification of Chevalley generators with inequivalent choice of the A_9 subalgebra:

$$E_a = K_{a+1}^a, \quad a = 1, \cdots, 8, \quad E_{11} = K_{10}^9, \quad E_9 = R_1^{910}, \quad E_{10} = R_2 \quad,$$

(64)

and corresponding Cartan subalgebra:

$$H_a = K_a^a - K_{a+1}^{a+1}, \quad a = 1, \cdots, 8, \qquad H_{11} = K_9^9 - K_{10}^{10}$$

$$H_9 = \quad K_9^9 + K_{10}^{10} + R_1 - \tfrac{1}{4}\sum_{a=1}^{10} K_a^a, \quad H_{10} = -2R_1 \quad . \tag{65}$$

As in the case of the mIIA theory, excluding the $\mathbf{R}_s$ generator of $GL(10, \mathbf{R})$ and the translations, it can be verified by inspection of the commutator algebra given in Eqs. (7) and (8), that this choice of Chevalley basis suffices to generate the remaining p-form generators of $\mathcal{G}_{IIB}$ algebra. We can define the nonlinear realization as before, identifying a corresponding group element for each of the $(p + 1)$-form generators in the IIB theory, coupling to supergravity pbranes. As shown by West [31], the analysis permits a successful deduction of the line elements of the full spectrum of type II branes: the p=1, 5 branes of the NS-NS sector and p=-1,1,3,5,7 Dbranes of the R-R sector. The NS-NS p=7 solution simply recovers Minkowskian spacetime.

Appendix C: Spacetime Reduction and Spacetime Emergence

Notice that from the perspective of E_{11} as illustrated in West's analysis in [31], there is nothing to distinguish 11d supergravity from the ten-dimensional IIA and IIB theories: all three share the same rank eleven symmetry algebra. To see that the latter two are *ten-dimensional* field theories requires that we introduce the notion of *spacetime*: in the purely algebraic formulation of $E_8^{(3)}$ proposed by West [24, 31], there are no spacetime translation generators, $\mathcal{P}_a$, to begin with.

There are three opposing suggestions for how to introduce spacetime into the formalism of nonlinear realizations [31]. In the context of $E_8^{(2)}$, also a competing proposal for the symmetry algebra of M theory [36], Damour, Henneaux, and Nicolai [33] proposed that the fields in the nonlinear realization should be taken as functions of time alone. Spatial dependence would arise thru the action of certain higher level generators of the $E_8^{(2)}$ algebra, which had the correct commutation relations to be identified as spatial derivatives. Thus, [33] showed that the 3-form, 6-form, and dual graviton representations appearing at level 1, 2, and 3, of $E_8^{(2)}$ contain tensors with the index structure of a kth spatial derivative at levels $1 + 3k$, $2 + 3k$, and $3 + 3k$. It is expected that similar identifications can be made for $E_8^{(3)}$ [35]. An alternative viewpoint has been put forward by West [25], namely, that spacetime can be incorporated by constructing the nonlinear realization of the semi-direct product of $E_8^{(3)}$ with a particular lowest weight representation denoted $\bar{l}_1$: the $\bar{l}_1$ representation would contain the coordinates of spacetime, but it should be noted that it also contains coordinates corresponding to the central charges of the supergravity theory. Details can be found in [35]. A third suggestion comes from Englert and Houart [34], who propose that the fields in the nonlinear realization depend upon an auxiliary parameter, which is extended to a full spacetime by

33

the identification of generators corresponding to spatial derivative operators in the very-extended algebra, analogous to the proposal of [33].

Since all of these proposals are at a preliminary stage of investigation, it behooves us to keep an open mind. On the other hand, it should be noted that our nonperturbative proposal for M Theory [1, 2] dovetails neatly with the algebraic formalism of hidden symmetries: identify the hidden symmetry algebra with the extended symmetry algebra of a reduced unitary matrix model that implements local symmetries. The large N limit of such a unitary matrix model naturally provides for the emergence of the coordinates of a spacetime continuum as shown in [2]. Let us review this important result.

We will begin by explaining our modified prescription for planar reduction such that one obtains non-trivial reduced matrix models from the planar reduction of gravity theories. Consider introducing a *flavor* quantum number in the nonmaximal 10d Einstein-Yang-Mills Lagrangian [18, 19, 21, 22, 2], replacing the gravitational spin-connection and Yang-Mills vector potential with $N{\times}N$ matrix arrays as follows:

$$
\begin{aligned}
\mathcal{R} \rightarrow \quad & \partial_a(\omega_b^{ab})_{AB} - \partial_b(\omega_a^{ab})_{AB} + (\omega_a^{ac})_{AC}(\omega_{ac}^b)_{CB} - (\omega_a^{ac})_{AC}(\omega_{bc}^b)_{CB} \\
F_{ab}^i \rightarrow \quad & \partial_a(A_b^i)_{AB} - \partial_b(A_a^i)_{AB} + f^{ijk}(A_a^j)_{AC}(A_b^k)_{CB} \quad ,
\end{aligned}
\tag{66}
$$

where the indices run from $i{=}1, \cdots,$ dim G, and $A, B{=}1, \cdots, N$. We will need to include a trace over the $U(N)$ flavor group in order that the new Lagrangian density transform as a $U(N)$ singlet. This Lagrangian will have the symmetry group $U(N){\times}G$, except that the $U(N)$ is *not* a gauge symmetry. Rather, it plays the role of a flavor group. How does one give meaning to the planar reduction of a locally supersymmetric Lagrangian with a *huge* flavor symmetry group to a single spacetime point? And why have we introduced a large N flavor symmetry, as opposed to the usual large N gauge symmetries invoked in [3, 5, 6]?

Notice that if we were to carry out the large N extension in analogy with the planar reductions of rigid Yang-Mills Lagrangians [3, 6], namely, replace the anomaly-free Yang-Mills group with the unitary large N group: $SO(32) \rightarrow U(N)$, where $SO(32) \subset U(32) \subset U(N)$, and where $U(N)$ is a fully *gauged* symmetry, we would find nothing of interest in the gravity sector of the Lagrangian. Suppressing all spacetime derivatives in the supergravity-Yang-Mills Lagrangian as usual, we obtain the standard quartic unitary matrix potential in the Yang-Mills sector: $[A_\mu, A_\nu]^2$, where A is now a matrix of rank N, but the Einstein sector yields an un-

interesting finite rank correction to these terms. Thus, since the zehnbein and spin connection are *finite*-dimensional matrix arrays when reduced to a single spacetime point, dimensional reduction of the Einstein action gives a *finite* number of terms of the general form, $E_a^\mu \omega_\mu^{ac} \omega_{c\lambda}^b E_b^\lambda$. It is clear that if one desires a nontrivial modification of large N dynamics of the Yang-Mills sector *the gravity variables must also scale with N*. This implies that the role of $U(N)$ in the continuum field theory Lagrangian must be that of a *flavor* symmetry group, rather than of a gauge group. We will find that the introduction of $U(N)$ as a flavor symmetry in the continuum Lagrangian enables the democratic appearance of large N scaling behavior in both the gauge and gravity sectors of the matrix model obtained upon spacetime reduction to a single point.

This motivates the first innovation introduced by us in [1, 2]: both the vector potential, zehnbein, and spin connection, were required to transform in the adjoint representation of a large N flavor group. The same requirement was made of the other bosonic fields in the Lagrangian, namely, the dilaton and two-form potential. What about the spinors in the supergravity Lagrangian? Since we have required that the large N flavor group commute with the group of supersymmetry transformations, it is important that fields which are partners under supersymmetry belong to the same $U(N)$ representation. Thus, we will require that all spinor fields gravitino, dilatino, and gaugino, also transform in the adjoint representation of the large N flavor group. Keeping only the terms in the continuum Lagrangian that remain after planar reduction, gives the following supermatrix Lagrangian:[21]

$$
\begin{aligned}
\mathcal{L}_{\text{planar}}^{(10d)} \quad &= \quad \tfrac{1}{2}\tfrac{1}{g^2}\text{tr}\left(f^{ijk}f^{ilm}A^{la}A^{mb}A_a^j A_b^k \;+\; \bar{\chi}^i\Gamma^a A_a^j \tau^j \chi^i\right) \\
&+ \tfrac{1}{\kappa^2}\text{tr}\left(E_a^\mu\left[\omega_\nu^{ac}\omega_{\mu c}^b - \omega_\mu^{ac}\omega_{\nu c}^b\right]E_b^\nu - A_a^i\tau^i\Phi A_a^j\tau^j\Phi + \bar{\chi}^i\Gamma^a\omega_{abc}\Gamma^{bc}\chi^i\right) \\
&+ \tfrac{1}{\kappa^2}\text{tr}\left(\bar{\psi}_a\Gamma^{abc}\omega_{bde}\Gamma^{de}\psi_c - 4\bar{\lambda}\Gamma^{ab}\omega_{ade}\Gamma^{de}\psi_b - 4\bar{\lambda}\Gamma^a\omega_{ade}\Gamma^{de}\lambda\right) \\
&+ \frac{4}{3}\frac{1}{g^4}\text{tr}\left(f^{nlm}f^{ijk}A^{n[a}A^{lb}A^{mc]}A_{[a}^i A_b^j A_{c]}^k\right) + \text{tr } \mathcal{L}_{2-\text{fermi}} \;+\; \text{tr } \mathcal{L}_{4-\text{fermi}} \quad (67)
\end{aligned}
$$

where $i, j, k, \cdots$ are group indices for the finite-dimensional Yang-Mills gauge group, and repeated indices are to be summed. The notation "tr" denotes, instead, the trace over the large N flavor group, whose indices have been suppressed. The first line of this expression is familiar: analogous terms appear in both the Banks-Fischler-Shenker-Susskind [5] and Ishibashi-Kawai-Kitazawa-Tsuchiya [6] rigid matrix models. The index structure of the terms in the first line make the $U(N){\times}G$

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[21]We denote the spacetime field, $f(x)$, and its planar-reduced representative which lives at the origin of spacetime, $f(0)$, by the same symbol f.

symmetry of the supermatrix model manifest: the model obtained by restricting to only the terms in the first line of this expression defines the *simplest* possible supermatrix model consistent with this symmetry group.

Thus, the distinction between flavor, and gauged, large N symmetry becomes significant only when we take into account the remaining terms in the matrix Lagrangian: we find new large N matrix variables originating in the supergravity sector of the continuum Lagrangian, as well as new multi-matrix interaction terms. These include a sixth-order self-interaction for the Yang-Mills potential, a term which was absent in both the BFSS and IKKT matrix models [5, 6], and which arises from the Chern-Simons contribution to the supergravity three-form field strength. It is evident that the symmetry structure of the full supermatrix Lagrangian given in Eq. (67) is much more subtle than simply $U(N) \times G$. Knowledge of the Cremmer-Julia hidden symmetries of the continuum field theory Lagrangian becomes a useful tool for its analysis.

It is illuminating to examine the form of matrix Lagrangian obtained by the planar reduction of the 11d supergravity theory. Recall the absence of Yang-Mills gauge fields, as well as the absence of a dilaton supermultiplet, in 11d supergravity. The 11d supergravity theory does, however, include a four-form field strength. The associated three-form potential couples to the supermembrane. Introduction of a large N flavor quantum number in the continuum Lagrangian, followed by spacetime reduction of the field theory to a single spacetime point, gives an elegant and especially simple matrix model:

$$\mathcal{L}_{\text{planar}}^{(11d)} = \frac{1}{\kappa^2} \text{tr} \left\{ \bar{\psi}_a \Gamma^{abc} \omega_{bde} \Gamma^{de} \psi_c + \omega_b^{ac} \omega_{ac}^b - \omega_a^{ac} \omega_{bc}^b \right\} \quad , \tag{68}$$

where the gravitino, ψ_a, is a 32-component Grassmann-valued array, and ω_{abc} is the spin-connection. Notice that the the presence of a higher p-form supergravity potential in the continuum field theory Lagrangian is, unfortunately, erased from the planar reduced matrix model: there is no analogous Chern-Simons coupling to a Yang-Mills field, as was present in Eq. (67). It may well be true that this particular supermatrix Lagrangian falls within the class of solvable zero-dimensional multi-matrix models, enabling a detailed analysis by well-established matrix model techniques. We should emphasize that this model is the precise, pure gravitational, supermatrix model analog of the planar reductions of rigid supersymmetric Yang-Mills theory considered in [3, 5, 6]. However, the model appears not to capture the full content of M theory because it lacks any knowledge of the crucial supermembrane sector of the theory.

Thus, while the planar reduction of gravity theories with large N flavor group has led to an interesting new class of zero-dimensional matrix models, these models appear not to capture the full content of M theory, inclusive of the crucial brane-spectrum required by duality. This brings us to a second innovation introduced in [1]. Notice that the Lagrangian considered in [21, 22, 1, 2] describes a ten-dimensional supergravity theory in generic curved spacetime background. Inherent in this expression is the notion of a local ten-dimensional flat tangent space attached to every point in spacetime. The naive procedure of planar reduction we have borrowed from rigid super-Yang-Mills theories [3] has ignored this aspect of the supergravity Lagrangian. We will now show that the spacetime reduction of all spacetime fields to *linear* forms defined on the infinitesimal patch of local tangent space at a single spacetime point, suffices to ensure that all of the local symmetries of the continuum Lagrangian are preserved in a corresponding reduced matrix model.

Our starting point is the unusual field theory Lagrangian with a huge flavor symmetry group: all fields, bosonic and fermionic, are required, in addition, to live in the adjoint representation of the large N unitary group, $U(N)$. We emphasize that $U(N)$ is a flavor symmetry; only the finite rank anomaly-free Yang-Mills group G has been gauged. Starting with the nonmaximal d=10 supergravity theory coupled to $O(32)$ Yang-Mills fields, we have the corresponding $U(N)$ invariant Lagrangian:

$$
\begin{aligned}
\mathcal{L} \quad &= \quad \tfrac{1}{\kappa^2}\mathrm{tr}\left\{\bar{\psi}_a\Gamma^{abc}D_b(\omega)\psi_c \;-\; 4\bar{\lambda}\Gamma^{ab}D_a(\omega)\psi_b \;-\; 4\bar{\lambda}\Gamma^a D_a(\omega)\lambda\right\} \\
&\quad + \tfrac{1}{g^2}\,\mathrm{tr}\left\{\bar{\chi}^i\Gamma^a D_a(\omega, A)\chi^i \;+\; \tfrac{1}{2}\,F^{ab}(A)F_{ab}(A)\right\} \\
&\quad\quad + \tfrac{1}{\kappa^2}\mathrm{tr}\left\{\mathcal{R}(\omega, E) \;-\; \partial^a\Phi\,\partial_a\Phi \;+\; 3H^{abc}H_{abc}\right\} \\
&\quad\quad\quad + \mathrm{tr}\left\{\mathcal{L}_{2-\mathrm{fermi}} \;+\; \mathcal{L}_{4-\mathrm{fermi}}\right\} \quad .
\end{aligned}
\tag{69}
$$

where the notation "tr" denotes taking the trace over the large N flavor group, and the two- and four-fermi terms are as given in [22, 1, 2]. Notice that each term in the Lagrangian is a flavor singlet, and the $U(N)$ flavor group commutes with *all* of the spacetime symmetries of the Lagrangian: namely, local Lorentz and local supersymmetry transformations, in addition to Yang-Mills gauge transformations.[22]

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[22]In our earlier papers [1], we have pointed out a more general possibility for the matrix superalgebra. Namely, the parameters for infinitesimal supersymmetry and $SL(n, \mathbf{R})$ transformations could themselves be non-singlet under the flavor $U(N)$. While we know of no reason to rule out such an extension, it is not necessary for the problem at hand. Notice that, for such matrix algebras, the large N limit would have to correspond to an exotic (nonlinear) extension of the Nahm classification of spacetime linear superalgebras. We thank Bernard de Witt for pointing this out.

The E_a^μ are the fundamental variables appearing in the matrix Lagrangian, but they are not all independent. Assuming a flat tangent space of Minkowskian signature, η_{ab}, the usual relation for the spacetime metric tensor takes the form of a $U(N)$ identity:

$$G^{\mu\nu} = \mathrm{tr}\left(E_a^\mu E_b^\nu\right), \quad \mu, \nu, \text{ and } a, b = 0, \cdots 9, \qquad E_a^\mu = G^{\mu\nu} E_{\nu a} \quad . \qquad (70)$$

As is familiar from differential geometry, $G^{\mu\nu}$ is the object that raises spacetime indices, while η^{ab} is the object that raises indices in tangent space. The spacetime metric transforms as a $U(N)$ singlet, as does η^{ab}. The usual constraint equation relating them is automatically satisfied:

$$\eta_{ab} = \mathrm{tr}\left(E_a^\mu E_{\mu b}\right) = G^{\mu\nu}\mathrm{tr}\left(E_{\nu a} E_{\mu b}\right) = G^{\mu\nu}\mathrm{tr}\left(G_{\nu\lambda}\eta_{ac}E^{\lambda c} E_{\mu b}\right) = \delta_\lambda^\mu \mathrm{tr}\left(\eta_{ac}E^{\lambda c} E_{\mu b}\right) = \eta_{ab} \quad .$$
$$(71)$$

We will now reduce all spacetime fields to *linear* forms defined on the infinitesimal patch of local tangent space at a single spacetime point as explained above. In other words, instead of simply setting all spacetime derivatives to zero as in the previous section, we retain the $O(\delta\xi^a)$ terms of the continuum Lagrangian, truncating at $O((\delta\xi^a)^2)$ in the Taylor expansion on tangent space, in the infinitesimal vicinity of the spacetime origin. We have parameterized the infinitesimal patch of local tangent space at the origin by the variables ξ^a, $a=0, \cdots, 9$. Recall the usual relation in Riemannian differential geometry linking the partial derivative operators acting in spacetime, and in the local tangent space:

$$\partial_\mu = E_\mu^a \partial_a, \qquad \mu, \nu = 0, \cdots, 9, \quad a, b = 0, \cdots, 9 \quad , \qquad (72)$$

Since the zehnbein is a flavor adjoint, an $N \times N$ dimensional array, whereas $\partial/\partial\xi^a$ is the ordinary partial derivative operator acting on a continuous and differentiable space with the local geometry of R^{10}, *it follows that the partial derivative operator in spacetime, ∂_μ, is also $U(N)$ valued.* In particular, consistency with the obvious identity $\partial_\mu X^\mu = 1$, implies that:

$$(X^\mu)_{AB} \equiv (E_a^\mu)_{AB}\delta\xi^a, \qquad (\partial_\mu)_{AB}(X^\mu)_{BC} = (\mathbf{1})_{AC}, \qquad A, B = 1, \cdots N \quad . \qquad (73)$$

In other words, the coordinate vector, X^μ, is itself $U(N)$ valued! Notice that X^μ is a dependent variable in our framework: it is *derived* from the zehnbein, E_μ^a, which is the fundamental variable appearing in the matrix Lagrangian. Unlike tangent space, which is smooth and differentiable, at least infinitesimally, spacetime contains a single element, a single spacetime "point". All variables defined at this

single spacetime point are $N \times N$ dimensional matrices, the fundamental degrees of freedom in the matrix model Lagrangian. Further, we will require of all forms on tangent space that they satisfy the *linearity* property: quadratic, and higher, derivatives, are identically set to zero, $\partial_n f(0) = 0$, $n \geq 2$, reflecting the fact that we are defining a set of functions on a base manifold of *infinitesimal* extent.

Why is there a need to retain an infinitesimal patch of tangent space while performing the dimensional reduction of the gravity theory to a single spacetime point? To develop some intuition into the $U(N)$ valued relations given above, notice that no restrictions have been placed upon the eigenvalue spectrum of the various zehnbein. In principle, one can solve for the eigenvalue spectrum of each E_a^μ, given the equation of motion that follows from the classical matrix Lagrangian. One of the solutions to the equation of motion corresponds to choosing the 10d Minkowskian flat space time metric as classical background:

$$< G^{\mu\nu} > \quad = \quad < \operatorname{tr} \left(E_a^\mu E_b^\nu \right) > \quad = \quad \eta^{\mu\nu}, \quad \mu, \nu, \text{ and } a, b = 0, \cdots 9 \quad . \tag{74}$$

We solve for the corresponding $< E_\mu^a >$, expressing them in diagonal form, and ordering the eigenvalues along the diagonal to reflect a *monotonic increase*. It is evident that in the large N limit, the eigenvalues will crowd together forming a continuum. Of course, as a consequence of the identity in Eq. (73), the coordinate matrices, $< X^\mu >$, also take diagonal form, their entries reflecting the monotonic increase along the diagonals of individual zehnbein. It is natural to interpret the ordered continuum of eigenvalues of the coordinate matrix as coordinate-locations for the continuum of spacetime points along the coordinate axis x^μ of 10d Minkowskian spacetime. Thus, we have recovered the coordinates of the spacetime continuum by taking the large N limit of the matrix model!

We are now ready to carry out the spacetime reduction of the Lagrangian given in Eq. (69) in accordance with our new prescription. Note that all fields, bosonic or fermionic, transform as adjoints under the flavor $U(N)$, and every term in the Lagrangian is a $U(N)$ singlet. The Lagrangian is manifestly invariant under local supersymmetry and local Lorentz transformations, and these symmetries commute with the flavor $U(N)$. Under spacetime reduction, every field in the Lagrangian is reduced to at most a *linear* form on the local tangent space, reflecting the fact that tangent space is an *infinitesimal* manifold. The sole exception is the zehnbein: since the spacetime coordinates have turned out to be in one-to-one correspondence with the eigenvalue spectrum of the zehnbein in the large N limit, self-consistency

requires that the reduction of the zehnbein is to a *zero-form* on tangent space. Most importantly, this also has the natural consequence that the local symmetries of the continuum Lagrangian can be made manifest in the reduced matrix model.

The remaining independent dynamical fields in the Lagrangian reduce to *linear* forms on tangent space. Our notation for a generic one-form $f(\xi)$ is as follows: $f(\xi) = f(0) + \partial_a f(0)\,\delta\xi^a$, where $\partial_a f(0)$ denotes, more precisely, the partial derivative of f with respect to ξ^a, evaluated at $\xi^a = 0$. Since every field in the continuum Lagrangian is an $N{\times}N$ array under flavor $U(N)$, and global symmetries are preserved under spacetime reduction, $f(0)$ and $\partial_a f(0)$ are two *independent* unitary matrices appearing in the matrix Lagrangian. Of course, one or other matrix array will be found to drop out of any given term in the Lagrangian. Thus, the matrix Lagrangian will turn out to have *exactly* the same symmetry group as the original continuum field theory Lagrangian. Listing each of the independent dynamical fields appearing in Eq. (69), we have the following result upon spacetime reduction to corresponding $N{\times}N$ matrix arrays defined at the origin $x = 0$, which we choose coincident with the origin of tangent space, $\xi = 0$:

$$
\begin{aligned}
E_a^\mu(x) &\rightarrow E_a^\mu(0) \\
A_a^i(x) &\rightarrow A_a^i(0) + \partial_b A_a^i(0)\delta\xi^b \\
\partial_c A_d^i(x) &\rightarrow \partial_c(A_d^i(0)) + \partial_b A_d^i(0)\partial_c(\delta\xi^b) = \partial_c A_d^i(0) \\
D_c A_d^i(x) &\rightarrow \partial_c A_d^i(0) - \partial_d A_c^i(0) + f^{ijk}A_c^j(0)A_d^k(0) \\
H_{abc}(x) &\rightarrow \partial_{[c}B_{ab]}(0) - g^2\left(A_{[a}^i(0)\partial_b A_{c]}^i(0) - \tfrac{2}{3}f^{ijk}A_{[a}^i(0)A_b^j(0)A_{c]}^k(0)\right) \\
D_a\Phi(x) &\rightarrow \partial_a\Phi(0) - A_a^i\tau^i\Phi(0) \quad,
\end{aligned}
\tag{75}
$$

where the indices have range as follows: $\mu = 0, \cdots, 9$, $a, b, c = 0, \cdots, 9$, and $i, j, k = 1, \cdots, \dim G$. We remind the reader that each of the objects on the left-hand-side of this list is also an $N{\times}N$ unitary matrix, the flavor indices have simply been suppressed. Suppressing the "(0)" dependence, we obtain the matrix Lagrangian:

$$
\begin{aligned}
\mathcal{L}^{(\mathrm{mat})} &= \tfrac{1}{\kappa^2}\mathrm{tr}\left\{\bar\psi_a\Gamma^{abc}D_b(\omega)\psi_c - 4\bar\lambda\Gamma^{ab}D_a(\omega)\psi_b - 4\bar\lambda\Gamma^a D_a(\omega)\lambda\right\} \\
&+ \tfrac{1}{\kappa^2}\mathrm{tr}\left\{\mathcal{R}(\omega, E) - \partial^a\Phi\,\partial_a\Phi + 3H^{abc}H_{abc}\right\} \\
&+ \tfrac{1}{g^2}\mathrm{tr}\left\{\tfrac{1}{2}\,F^{ab}(A)F_{ab}(A) + \bar\chi^i\Gamma^a D_a(\omega, A)\chi^i\right\} \\
&+ \mathcal{L}_{2-\mathrm{fermi}}^{(\mathrm{mat})} + \mathcal{L}_{4-\mathrm{fermi}}^{(\mathrm{mat})} \quad.
\end{aligned}
\tag{76}
$$

In other words, the matrix Lagrangian takes precisely the *same* form as the original continuum Lagrangian with large N flavor group, except that all spacetime fields are restricted to their value at the origin: the infinite number of degrees of freedom

in the original continuum field theory have indeed been drastically thinned to those of a zero-dimensional matrix model with $U(N)$ flavor symmetry. But, remarkably, by the introduction of infinitesimal linear forms on the local flat tangent space, this matrix Lagrangian also preserves a remnant of the *local* symmetries of the continuum Lagrangian. The underlying reason why there exists a matrix Lagrangian that can make manifest the local symmetries of a given continuum field theory, is the spacetime *locality* property of the Lagrangian density in quantum field theory. Further details, including the crucial two- and four-fermi terms required by closure of the supersymmetry algebra can be found in [1, 2], following [21, 22].

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